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Cooperation in Controlled Network Structures

Vincent Feltkamp

Cooperation in Controlled Network Structures

Proefschrift

ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof. dr. L.F.W. de Klerk, in het openbaar te verdedigen ten overstaan van een door het college van dekanen aangewezen commissie in de aula van de Universiteit op

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door

Vincent Feltkamp

geboren op 25 juni 1967 te Ciney

PROMOTOR : Prof. Dr. S. H. Tijs
COPROMOTOR : Dr. P.E.M. Borm

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Contents

Take seven and a half,	קח ז' וחצי,
and leave us.	ולך מעמנו
THE BOOK OF THE NUMBER	ספר המספר
Abraham Ibn Ezra (1146)	אברהם אבן עזרה

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Chapter 1

Introduction

Though its name suggests a more playful topic, game theory studies decision problems featuring conflict and cooperation between the decision makers. The name is derived from the paper 'Zur Theorie der Gesellschaftsspiele' by von Neumann (1928). After publication of this paper, game theory remained an obscure subject for 16 years, until the publication in 1944 of the book 'Theory of games and economic behavior' by von Neumann and Morgenstern kindled intensive game theoretic research.

Game theory is usually divided into two parts, cooperative and non-cooperative game theory. Non-cooperative game theory analyzes which actions the decision makers will take, if they cannot make binding agreements. Cooperative game theory, on the other hand, assumes that before decisions take place, players can make binding agreements about their decisions. Attention is then concentrated on the questions 'which coalition will emerge?' and 'how should the profits derived from cooperation be distributed?'

Example 1.0.1 Consider Ann, Bart and Charley, whose homes are not yet connected to running water. Each one wants her/his house to be connected via a water pipe to the spring, in order to avoid carrying water every day. It is not necessary for each one to be connected directly to the spring; being connected via others is sufficient. Assuming that the pipes are large enough for three players, one pipe can serve more than one person. The situation is sketched schematically in figure 1.1. The costs of the pipes are as indicated. If everybody is directly connected to the spring, total costs are $40 + 50 + 60 = 150$, whereas if Ann is directly connected to the spring, Charley is connected via Ann, and Bart via Charley and Ann, the total costs can be lowered to $40+26+35 = 101$. Lowering the costs further is impossible. Hence, there is a profit of 49 if Ann, Bart and Charley cooperate. Of course, if they do cooperate, they will bargain about how to share this profit. A good allocation of the profit should take into account how much subgroups can achieve. For example, Ann and Charley can argue that together they can achieve a profit of $100 - 66 = 34$ by connecting Charley via Ann instead of directly to the spring. Similarly, Ann and Bart can achieve a profit of $90 - 85 = 5$ and Bart and Charley can achieve a profit of $110 - 85 = 25$. Hence a good allocation should allocate the total profit 49 in such way that Ann and Charley together get at least 34, Ann and Bart jointly get

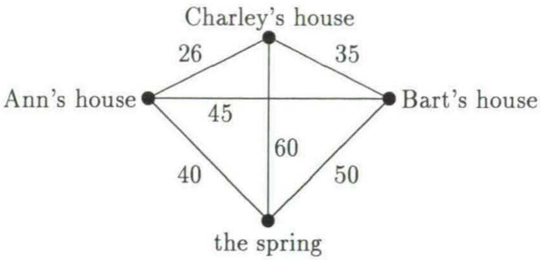


Figure 1.1: A connection problem.

at least 5 and Bart and Charley jointly receive at least 25.

This example shows that the economic possibilities of a group of agents are determined by a network. A second example highlights a situation in which producers exercising joint control over resources can profit from cooperation.

Example 1.0.2 David, a farmer growing mainly hops, and Ed and Flora, a couple jointly owning a farm that is good in growing barley, brew beer and decide to start a beer brewing cooperative. The cooperative will work as follows. Ed and Flora have a lager brewery, David has an ale brewery. In the cooperative, it is agreed that these breweries can be used by any brewer. However, the resources (hops and barley) are privately owned. Both hops and barley (as well as water and yeast, which are supposed to be easy to get) are needed to brew beer. Hence, the farmers can by cooperating increase their revenue.

Suppose Ed and Flora have 26 acres of barley and one acre of hops while David has four acres of barley and 11 acres of hops and brewing 100 hogsheads of ale (lager) uses five (four) acres of barley and two (three) acres of hops, respectively. Suppose both ale and lager earn a profit of \$100 per hogshead, net of all costs. If David, Ed and Flora cooperate, they optimally produce 300 hogsheads of lager and 450 hogsheads of ale, obtaining a revenue of $\$75,000$. As in the previous example, an allocation of this revenue to the players should take the outside options of each player into account. For example, Ed by himself cannot produce anything—as he would need the fiat of Flora; vice versa, Flora by herself cannot brew anything, either. Hence, each can guarantee a zero profit and will not accept a negative payoff. Together, the couple can produce 50 hogsheads of ale (at David's farm), which earns them $\$5,000$. Similarly David can brew 100 hogsheads of lager, earning himself $\$10,000$.

In this monograph, which consists of two more or less independent parts, I use game theory to analyze cooperation in economic situations involving network structures, of which parts are controlled by the agents.

Part I explores situations related to example 1.0.1, in which a group of users has to be connected as cheaply as possible to a supplier of a service. Because the cost of a network connecting all users to the supplier has to be borne by the users, the problem of allocating this cost is addressed. In the literature the network construction problem and the cost allocation problem are usually treated separately but as they are two facets of one problem I will consider them together. I will mainly address the cost allocation problem with methods from cooperative game theory and concentrate on the core of associated games.

It will be shown that Bird's tree allocations are closely related to Prim-Dijkstra's algorithm for constructing a minimum-cost network that connects all users to the supplier. By considering the solution to such a connection problem to consist of a network together with a cost allocation it is possible to axiomatically characterize the Bird tree rule.

Associating Bird's tree allocations with Prim-Dijkstra's algorithm leads to the question whether other allocation rules can be derived from the other algorithms that construct minimum-cost networks, viz. Kruskal's algorithm and Borůvka's algorithm. I will show that the irreducible core is closely related to Kruskal's algorithm and will propose two other cost allocation rules associated with Kruskal's algorithm. Both these rules and the irreducible core are then axiomatically characterized. A cost allocation associated with Borůvka's algorithm will also be provided. All these cost allocations turn out to be refinements of the core of the associated games.

Furthermore, the classical model is extended to model also those situations in which an existing partial network has to be extended in order to connect every user to the supplier.

Moreover, I will present a non-cooperative approach in which each player strategically connects himself to the supplier and will prove that the Nash equilibria of these games coincide with Bird's tree allocations.

Finally, I will present an overview of alternative network construction models, such as a model in which users have to be connected to more than one source, a model in which users can choose whether to get connected to the source or to use a local source and a model in which the cost of a connection depends on the number of users of this connection. These models provide ideas for future research.

Part II is concerned with the influence of control exercised by players over economic resources on the division of the revenue generated by cooperation, as in example 1.0.2. The problems considered typically involve an economic situation that can generate profits and of which players control the resources. These 'resources' can take many forms, like parts of a network, primary goods of a production economy, parcels of land, etc.

The question addressed is how to allocate the profits generated by cooperation among the players in a way that takes account not only of the profits but also of the control structure. As before, these situations are modeled by means of cooperative games, and I am interested in allocations of the revenue accrued by cooperation, which are derived from game-theoretic solutions, such as the Shapley value, the core or population monotonic allocation schemes.

The control that players exercise over the resources is modeled by means of control games. These are simple games in which the grand coalition is winning. In order to get a better understanding of the influence of the control I will systematically investigate properties of simple games. This investigation leads to alternative axiomatic characterizations of the Shapley value on the class of control games, the class of simple games and the class of all cooperative games. In a similar way, the Banzhaf value is also axiomatically characterized on these classes of games.

Finally, I will provide a fast algorithm to compute the nucleolus and kernel of veto-rich games, in which there is a player whose absence prevents the others from obtaining a positive payoff. These games are generalizations of balanced simple games and occur, for example, in markets with a monopolist or monopsonist.

Preliminaries and notations

A cooperative *Transferable Utility game* (TU-game) (N, v) consists of a (finite) set N of *players* and a *characteristic function* v which assigns to each subgroup or *coalition* S of players a real number that is to be interpreted as the maximal gains or minimal cost this coalition can guarantee by cooperating, regardless of the actions of the other players. It is assumed that $v(\emptyset) = 0$; i.e. the empty set cannot achieve anything. Often a game (N, v) is identified with its characteristic function v. The *power set* of a set N will be denoted by 2^N.

The cardinality of a set S will be denoted by $|S|$. If $x \in \mathbf{R}^N$ and $S \subseteq N$, the following notation will be used :

$$x(S) := \sum_{i \in S} x_i.$$

I will recall some definitions to show the notational conventions of graph theory used in this monograph. These can also be found in any elementary textbook on graph theory (e.g. Wilson (1972)). An (undirected) graph $\langle V, E\rangle$ consists of a set V of vertices and a set E of edges. Each edge connects two vertices, and is said to be incident with these vertices. An edge e incident with vertices i and j is identified with $\{i, j\}$[1]. For a graph $\langle V, E\rangle$ and a set $W \subseteq V$,

$$E(W) := \{e \in E \mid e \subseteq W\}$$

is the set of edges linking two vertices in W. For a set $E' \subseteq E$,

$$V(E') := \{v \in V \mid \text{there exists an edge } e \in E' \text{ with } v \in e\}$$

is the set of vertices incident with E'.

The complete graph on a vertex set V is the graph $K_V = \langle V, E_V\rangle$, where

$$E_V := \{\{v, w\} \mid v, w \in V \text{ and } v \neq w\}.$$

[1]Because multigraphs are not considered : two vertices are connected by at most one edge.

A *path* from i to j in a graph $\langle V, E\rangle$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $l \leq k$, the edge $\{i_{l-1}, i_l\}$ lies in E. A *cycle* is a path of which the begin point coincides with the end point.

Two vertices $i, j \in V$ are *connected* in a graph $\langle V, E\rangle$ if there is a path from i to j in $\langle V, E\rangle$. A subset W of V is *connected* in $\langle V, E\rangle$ if every two vertices $i, j \in W$ are connected in the subgraph $\langle W, E(W)\rangle$. A connected set W is a *(connected) component* of the graph $\langle V, E\rangle$ if no superset of W is connected. If $W \subseteq V$, the set of components of the graph $\langle W, E(W)\rangle$ is denoted W/E. A component of a graph will be denoted by the letter C, and for a vertex v of the graph, the component containing v is denoted C_v.

A *connected graph* is a graph $\langle V, E\rangle$ with V connected in $\langle V, E\rangle$. A *tree* is a connected graph without cycles. A *forest* is a graph without cycles. A *leaf* of a graph is a vertex that is incident to only one edge of the graph.

Part I

Network Construction and Cost Allocation

Consider a group of villages, each of which needs to be connected directly or via other villages to a source. Such a connection needs costly links. Each village could connect itself directly to the source, but by cooperating, each might reduce costs. This cost-minimization problem is an old problem in Operations Research, and Borůvka (1926) came up with algorithms to construct a tree connecting every village to the source with minimal total cost. Later, Kruskal (1956), Prim (1957) and Dijkstra (1959) found similar algorithms. A historic overview of this *minimum-cost spanning tree problem* can be found in Graham and Hell (1985).

However, constructing a minimum-cost spanning tree (*mcst*) is only part of the problem : if the villages must bear the cost of this tree, then a cost-allocation problem has to be addressed as well. Claus and Kleitman (1973) introduced this cost-allocation problem, whereupon Bird (1976) treated this problem with game-theoretic methods and proposed a cost-allocation rule that associates with each minimum-cost spanning tree a cost allocation. These allocations are known as *Bird's tree allocations.* As more than one mcst can exist for a given problem, Bird's rule can yield more than one allocation. Generically, however, only one mcst exists and then this rule yields a unique allocation.

Granot and Huberman (1981) proved that Bird's tree allocations are extremal points of the core of the associated minimum-cost spanning tree game. This game is defined as follows : the players are the villages and the worth of a coalition is the minimal cost of connecting this coalition to the source via links between members of this coalition. Not being satisfied with only one extremal point of the core, Granot and Huberman then provide the weak and strong demand operations, which yield more core elements when applied to Bird's tree allocations. Granot and Huberman's reason for looking at other core allocations than those obtained by Bird's rule, is that although core elements are stable against defection by subcoalitions, an extremal point of the core discriminates against some players. For example, Bird's tree allocations discriminate against the players closest to the root. Granot and Huberman's demand operations remedy this problem by allowing a player to demand contributions from players that are connected to the source via this player. Aarts (1992, 1994) found other extreme points of the core in case the mcst problem has an mcst that is a *chain*, i.e. a tree with only two leaves. Kuipers (1993, 1994) computed all extreme elements of the core of information graph games. These are games arising from mcst situations in which the costs of links are either one or zero. Furthermore, he provided an efficient algorithm for the nucleolus of these information graph games.

Other related network construction games are *Steiner tree games* (Megiddo, 1978 and Skorin-Kapov, 1994), *fixed-cost spanning forest games* (Granot and Granot, 1992), *capacitated network design games* (Skorin-Kapov and Beltran, 1993), *spanning network games* (Granot and Maschler, 1991 and Van den Nouweland, Maschler and Tijs, 1993). For an overview of network models in economics, see Sharkey (1993).

Chapter 2 provides alternative points of view of Bird's tree allocations : an axiomatic characterization of the set of Bird's tree allocations, and a non-cooperative game, in which Bird's tree allocations correspond to Nash equilibria.

Moreover, the network construction problem and the cost allocation problem are treated simultaneously. One reason is that they are two sides of the same problem, and solving one side gives insight into the other side. For example, examining Bird's tree-allocation rule for minimum-cost spanning tree problems, one sees that it is intimately related to the algorithm for finding minimum-cost spanning trees that is described in Prim (1957) and Dijkstra (1959). Here, Bird's tree-allocation rule is integrated into Prim and Dijkstra's algorithm.

This suggests allocation rules that correspond to the other algorithms for finding minimum-cost spanning trees, viz. the algorithm of Kruskal (1956), and the decentralized algorithm that was first described in Borůvka (1926). Chapters 3 and 4 elaborate on this approach.

Chapter 3 shows that the irreducible core, which was introduced in Bird (1976), can be obtained by an allocation rule closely related to Kruskal's algorithm. It turns out that all allocation rules mentioned in chapters 2, 3 and 4 are refinements of the irreducible core.

Chapter 4 proposes two one-point allocation rules, viz. the proportional and the decentralized rule, and axiomatically characterizes the proportional rule. This rule is closely related to Kruskal's algorithm for finding minimum-cost spanning trees, while the decentralized rule is related to Borůvka's algorithm.

Moreover, chapters 3 and 4 extend the class of problems to include problems in which a network is initially present. These problems are called *minimum-cost spanning extension (mcse)* problems. This extension of the class of problems is motivated by the consideration that a minimum-cost spanning tree (or extension) problem that is half solved can now be reconsidered as a minimum-cost spanning extension problem, after which the solution given for the original problem and the continuation problem can be compared.

Chapter 5 presents other network construction models. Rather than attempting to provide an exhaustive treatment of these models, my aim is to provide suggestions for future research. The first model assumes that players *need not* be connected to the source, but would like to, if this improves their well-being. The second model assumes that more than one source exists, but that the sources are unreliable; players must thus be connected to more than one source. The third model assumes that the cost of a connection depends on the number of players using it. The fourth section presents a non-cooperative game associated with network construction problems.

First follow a few standard definitions.

With many economic situations in which costs have to be divided one can associate a TU *cost game* (N, c) consisting of a finite set N of players, and a *characteristic function* $c : 2^N \to \mathbf{R}$, satisfying $c(\emptyset) = 0$. Here, $c(S)$ represents the minimal-cost for coalition S if it secedes, i.e. if people of S cooperate and cannot count on help from people outside S.

The *core* of a game consists of those allocations in which the worth of the grand coalition is distributed among the players in such a way that no coalition can improve its situation by seceding. For cost games, this translates into

$$\mathrm{Core}(c) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

A theorem by Bondareva (1963) and independently by Shapley (1967) states that a game has a non-empty core if and only if it is *balanced.* As in part I the condition of balancedness is never explicitly used, it will not be defined here. I will, however, call a game with a non-empty core balanced.

The economic situations in part I involve a set N of users of a source denoted $*$. For a coalition $S \subseteq N$, denote $S \cup \{*\}$ by S^*. Furthermore, for a vector $x \in \mathbf{R}^N$ and a player $i \in N$, denote by x^{-i} the restriction of x to $N \setminus \{i\}$.

For two vectors $x \in \mathbf{R}^S$ and $y \in \mathbf{R}^T$, where S and T are two disjoint coalitions, denote by $(x, y) \in \mathbf{R}^{S \cup T}$ the vector with coordinates

$$(x,y)_k := \begin{cases} x_k & \text{if } k \in S, \\ y_k & \text{if } k \in T. \end{cases}$$

Furthermore, for $S \subseteq T$ and a vector $x \in \mathbf{R}^T$, denote by x^S the restriction of x to S. For a coalition $S \subseteq N$, the symbol 1_S is used to denote the vector in $\mathbf{R}^N$ with coordinates

$$(1_S)_k := \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{if } k \in N \setminus S. \end{cases}$$

For any coalition S, the simplex Δ^S is defined by

$$\Delta^S := \{x \in \mathbf{R}^S_+ \mid \sum_{i \in S} x_i = 1\}.$$

Chapter 2

Minimum-cost spanning tree problems

Bird (1976) was the first to give an allocation rule for minimum-cost spanning tree problems, of which Granot and Huberman proved it generates core elements on the associated minimum-cost spanning tree games, thereby proving that these games are balanced. However, Bird's tree allocations have been criticized because they are extreme elements of the core.

This chapter is based on Feltkamp, Tijs and Muto (1994a). Section 2.1 recalls a few definitions and formally presents minimum-cost spanning tree problems and Bird's tree-allocation rule. Section 2.2 axiomatically characterizes the set of minimum-cost spanning trees together with the corresponding Bird tree allocations, using efficiency, leaf consistency and converse leaf consistency. Section 2.3 presents a non-cooperative game, in which a strategy of a player consists of choosing which (if any) edge to pay. It is shown that if all costs are positive, Bird's tree allocations coincide with the Nash equilibria of this game.

2.1 Bird's tree-allocation rule

A *minimum-cost spanning tree (mcst) problem* $\langle N, *, w \rangle$ consists of a finite group N of agents, each of whom wants to be connected to a common source, denoted by $*$. The non-negative cost of constructing a link $\{i, j\}$ between the vertices i and j in $N^* \equiv N \cup \{*\}$ is denoted by $w(i, j)$. Because of these costs, agents have an incentive to cooperate, and to construct a minimal cost graph that connects them all to the source. If a cycle appears in such a *minimum-cost spanning graph*, at least one edge in this cycle can be eliminated, which will yield a minimum-cost spanning graph with less cycles. Hence, minimum-cost spanning graphs exist that contain no cycles at all, i.e. they are trees. This explains the name of the problem.

Note that we implicitly assume that all edges can, in principle, be constructed, but it is possible to model a problem in which some edges cannot be constructed by making the cost of these edges very large.

Prim (1957) and Dijkstra (1959) proposed the following algorithm to find a minimum-cost spanning tree given an mcst problem.

Algorithm 2.1.1 (Prim and Dijkstra)

input : an mcst problem $\mathcal{T} \equiv \langle N, *, w \rangle$

output : the edge set T of a minimum-cost spanning tree

1. Choose a vertex $v \in N^*$ as *root*.
2. Initialize $T = \emptyset$.
3. Find a minimal cost edge $e \in E_{N^*} \setminus T$ incident to $\{v\} \cup N^*(T)$ such that joining e to T does not introduce a cycle. (Remember E_{N^*} are the edges of the complete graph on N^* and $N^*(T)$ are the vertices incident with the edges in T.
4. Join e to T.
5. If not all vertices are connected to the root in the graph $\langle N^*, T \rangle$, go back to stage 3.

Prim and Dijkstra prove that any graph resulting from the algorithm is an mcst and that by varying between the possible edges in step 3, this algorithm can construct all minimum-cost spanning trees of this mcst problem $\mathcal{T}$.

A closely related problem is how to allocate the cost of the edges of a minimum-cost spanning tree among the agents (users of the source) in a reasonable way. Bird (1976) proposed a cost-allocation rule for the mcst problem, which we call *Bird's tree-allocation rule*, because it associates a cost allocation to every mcst of the mcst problem. Given an mcst problem $\langle N, *, w \rangle$ and a mcst $\langle N^*, T \rangle$ for the grand coalition, Bird's tree allocation β^T is constructed by assigning to a player $i \in N$ the cost of the first edge on the unique path in the tree $\langle N^*, T \rangle$ from player i to the source $*$. In fact, this allocation is intimately linked with the Prim-Dijkstra algorithm : the tree $\langle N^*, T \rangle$ and the allocation β^T can be constructed together by choosing the source as root and allocating the cost of the edge added at a certain stage to the person that this edge newly connects to the the source. More formally, the algorithm is the following.

Algorithm 2.1.2 (Bird's rule integrated into Prim-Dijkstra's algorithm)

input : an mcst problem $\langle N, *, w \rangle$

output : an edge set T of an mcst and an allocation x (Bird's tree allocation β^T)

1. Choose the source $*$ as root.
2. Initialize $T = \emptyset$.
3. Find a minimal cost edge $e = \{i, j\} \in E_{N^*} \setminus T$ incident to $\{*\} \cup N^*(T)$ such that joining e to T does not introduce a cycle.

4. One of i and j, say j, was previously connected to the source and the other vertex, i, is a player that was not yet connected to the source. Assign the cost $x_i := w(e)$ to agent i.

5. Join e to T.

6. If not all vertices are connected to the root in the graph $\langle N^*, T\rangle$, go back to stage 3.

As the set of all trees obtained by Prim and Dijkstra's algorithm is independent of the root that is chosen, this algorithm yields the same trees as Prim and Dijkstra's algorithm, and for each tree $\langle N^*, T\rangle$, it yields an allocation that is Bird's tree allocation β^T associated with this tree. This is easy to see : in step 4, the edge e is precisely the first edge on the unique path from agent i to the source in the tree that will be constructed. If the mcst problem contains two or more edges with the same weight, more than one mcst might exist, and for a particular mcst $\langle N^*, T\rangle$, it could happen that there is more than one order in which Prim-Dijkstra's algorithm can choose the edges in T. Obviously, the order does not change the edge that a player has to pay according to Bird's tree allocation rule; Bird's tree allocation β^T is thus independent of the *order* in which the edges of the tree $\langle N^*, T\rangle$ are chosen. It does, however, depend on which tree is constructed. See example 2.1.3.

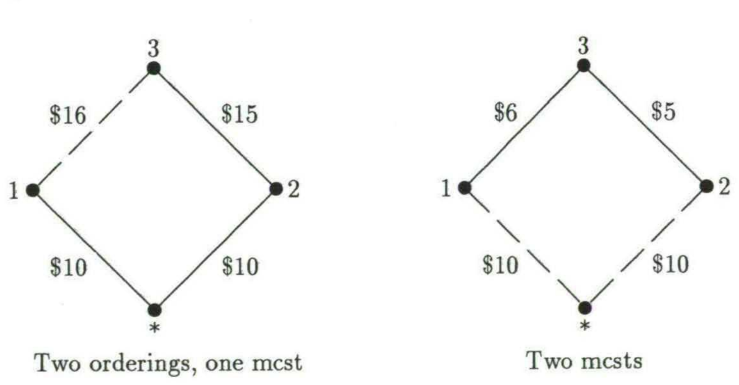


Figure 2.1: Edges that are not indicated cost $100.

Example 2.1.3 In the problem on the left-hand side of figure 2.1, it does not matter whether Prim-Dijkstra's algorithm chooses the links in the unique mcst in the order $(\{*,1\}, \{*,2\}, \{2,3\})$ or $(\{*,2\}, \{*,1\}, \{2,3\})$; the link $\{*,1\}$ is in both cases paid by player 1 and the link $\{*,2\}$ is in both cases paid by player 2.

In the problem on the right-hand side of figure 2.1, only one of the two dashed links will be constructed. If $\{*,1\}$ is constructed, Bird's tree allocation is $(10,5,6)$, and if $\{*,2\}$ is constructed, Bird's tree allocation is $(6,10,5)$.

Bird (1976) associated the following transferable utility mcst game $(N, c^{\mathcal{T}})$ to an mcst problem $\mathcal{T}$. The players are the agents and the worth $c^{\mathcal{T}}(S)$ of a coalition S is the minimal cost of a tree on $S^* := S \cup \{*\}$. In formula,

$$c^{\mathcal{T}}(S) = \min\{\sum_{e \in T} w(e) \mid T \subseteq E_S \text{ and } \langle S^*, T\rangle \text{ is a tree}\}$$

for all $S \subseteq N$. Granot and Huberman (1981) proved that Bird's tree-allocation rule yields extreme points of the core of the mcst game.

2.2 An axiomatic characterization of the Bird rule

This section characterizes the set of mcsts and their associated Bird allocations axiomatically, using efficiency, leaf consistency and converse leaf consistency.

A *solution of mcst problems* is a function ψ which assigns the following set :

$$\psi(\mathcal{T}) \subseteq \{((e^1, \ldots, e^\tau), x) \mid \langle N^*, \{e^1, \ldots, e^\tau\}\rangle \text{ is connected and } \sum_{i \in N} x_i \geq \sum_{t=1}^{\tau} w(e^t)\}$$

of edge sets of graphs and associated cost allocations to every mcst problem $\mathcal{T} = \langle N, *, w\rangle$. We mention a few properties of solutions of mcst problems.

Definition 2.2.1

NE A solution ψ is called *non-empty* if

$$\psi(\mathcal{T}) \neq \emptyset \qquad \text{for all mcst problems } \mathcal{T}.$$

Eff ψ is *efficient* if for all mcst problems $\mathcal{T}$, all $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T})$ are *efficient*, that is, for all $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T})$, $\langle N^*, \{e^1, \ldots, e^\tau\}\rangle$ is a minimal cost spanning tree and

$$\sum_{i \in N} x_i = \sum_{t=1}^{\tau} w(e^t).$$

The next two properties, leaf consistency and converse leaf consistency, give relations between solutions of a mcst problem and solutions of reduced mcst problems. A reduced mcst problem is an mcst problem in which some players have been eliminated. The idea is that solving reduced problems is easier than solving the original problem, and that the solution of the original problem should be related to the solution of the reduced problems. We only require this relation if one player that is a leaf in the graph of a proposed solution element is deleted, however. (A leaf of a graph is a vertex that has at most one incident edge.) The idea is that a leaf is not needed by any other player to get connected to the source; if a leaf player is missing, thus, this should not affect the other players.

Definition 2.2.2 Given an mcst problem $\mathcal{T} \equiv \langle N, *, w\rangle$ and a player $i \in N$, define the *reduced mcst problem* $\mathcal{T}^{-i}$ by

$$\mathcal{T}^{-i} := \langle N \setminus \{i\}, *, w^{-i}\rangle,$$

where w^{-i} equals the restriction of w to $E_{N^{*}\setminus\{i\}}$.

Note that the reduced problem does not depend on any fixed solution. Note also that it is indeed an mcst problem. Use reduced mcst problems to define leaf consistency and converse leaf consistency as follows :

Definition 2.2.3

LCons A solution ψ of mcst problems is *leaf consistent* if for every mcst problem $\mathcal{T}$, for every $((e^1,\dots,e^\tau), x) \in \psi(\mathcal{T})$ and for every player i that is a leaf in the graph $\langle N^*, \{e^1,\dots,e^\tau\}\rangle$,

$$((e^1,\dots,e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i}),$$

where $(e^1,\dots,e^\tau)^{-i}$ is obtained from $(e^1,\dots,e^\tau)$ by deleting the unique edge incident to i and x^{-i} is the vector obtained from x by deleting the coordinate of player i.

CoLCons A solution ψ of mcst problems is *converse leaf consistent* if for any mcst problem $\mathcal{T}$ and for any $((e^1,\dots,e^\tau), x)$ efficient in $\mathcal{T}$, the following is satisfied : if

$$((e^1,\dots,e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i})$$

for all players i that are leaves of $\langle N^*, \{e^1,\dots,e^\tau\}\rangle$, then

$$((e^1,\dots,e^\tau), x) \in \psi(\mathcal{T}).$$

The leaf-consistency property is motivated by the idea that an element of a solution, when a leaf is eliminated, should be an element of the solution to the reduced problem. The converse leaf-consistency property is motivated by the opposite idea, that no efficient candidates for a solution should be excluded, unless the 'reduced' solution is excluded as solution of a reduced problem in which a leaf has been deleted.

We use the properties mentioned to axiomatically characterize the Bird rule.

Definition 2.2.4 The Bird rule of an mcst problem $\mathcal{T}$ is the set

$$\beta(\mathcal{T}) := \{((e^1,\dots,e^\tau), \beta^T(\mathcal{T})) \mid T = \{e^1,\dots,e^\tau\} \text{ and } \langle N^*, T\rangle \text{ is an mcst of } \mathcal{T}\}$$

of sequences of edges of minimum-cost spanning trees and the corresponding Bird tree allocations.

Proposition 2.2.5 The Bird rule satisfies NE, Eff, LCons and CoLCons.

Proof : Efficiency was proven by Bird, and non-emptiness is evident. In order to prove LCons, assume $((e^1,\ldots,e^\tau),\beta^T) \in \beta(\mathcal{T})$ and let player i be a leaf in the tree $\langle N^*,T\rangle$, where $T = \{e^1,\ldots,e^\tau\}$. Define e to be the first edge on the unique path in $\langle N^*,T\rangle$ from i to the source. Then $(e^1,\ldots,e^\tau)^{-i}$ is obtained from $(e^1,\ldots,e^\tau)$ by deleting the edge e and is a sequence obtained by applying the Prim-Dijkstra algorithm to the reduced mcst problem $\mathcal{T}^{-i}$. Hence $x^{-i} = \beta^{T\setminus\{e\}}$, and

$$((e^1,\ldots,e^\tau)^{-i},x^{-i}) \in \beta(\mathcal{T}^{-i}).$$

In order to prove that the Bird rule satisfies CoLCons, assume that $((e^1,\ldots,e^\tau),x)$ is efficient in an mcst problem $\mathcal{T}$ and assume that player i is a leaf of $\langle N^*,\{e^1,\ldots,e^\tau\}\rangle$ such that

$$((e^1,\ldots,e^\tau)^{-i},x^{-i}) \in \beta(\mathcal{T}^{-i}). \tag{2.2.1}$$

Define e_i to be the unique edge incident to i in $\{e^1,\ldots,e^\tau\}$. Then $\{e^1,\ldots,e^\tau\} = \{e^1,\ldots,e^\tau\}^{-i}\cup\{e_i\}$ and $\langle N^*\setminus\{i\},\{e^1,\ldots,e^\tau\}^{-i}\rangle$ is an mcst for the reduced mcst problem $\mathcal{T}^{-i}$. Hence efficiency of $((e^1,\ldots,e^\tau),x)$ and equation 2.2.1 imply

$$\sum_{k\in N} x_k = \sum_{e\in\{e^1,\ldots,e^\tau\}} w(e) = \sum_{e\in\{e^1,\ldots,e^\tau\}^{-i}} w(e) + w(e_i) = \sum_{k\in N\setminus\{i\}} x_k^{-i} + w(e_i),$$

which implies that $x_i = w(e_i)$. So $((e^1,\ldots,e^\tau),x) \in \beta(\mathcal{T})$. □

Lemma 2.2.6 If a solution ϕ satisfies Eff and LCons, and a solution ψ satisfies NE, Eff and CoLCons, then $\phi(\mathcal{T}) \subseteq \psi(\mathcal{T})$ for all mcst problems $\mathcal{T}$.

Proof : Proceed by induction on the cardinality of N. Let $|N| = 1$ and denote by e the edge between the unique player and the source. By efficiency of both solutions and non-emptiness of ψ, obtain $\phi(\mathcal{T}) \subseteq \{((e),w(e))\} = \psi(\mathcal{T})$. Take an mcst problem $\mathcal{T}$ with $k > 1$ players, and suppose that for all mcst problems $\mathcal{T}'$ with less than k players, $\phi(\mathcal{T}') \subseteq \psi(\mathcal{T}')$. Take $((e^1,\ldots,e^\tau),x) \in \phi(\mathcal{T})$ and choose a leaf $i \neq *$ of the tree T induced by $(e^1,\ldots,e^\tau)$. Then by leaf consistency of ϕ, $((e^1,\ldots,e^\tau)^{-i},x^{-i}) \in \phi(\mathcal{T}^{-i}) \subseteq \psi(\mathcal{T}^{-i})$. Now, because $((e^1,\ldots,e^\tau),x)$ is efficient, converse leaf consistency of ψ implies $((e^1,\ldots,e^\tau),x) \in \psi(\mathcal{T})$. □

Theorem 2.2.7 The unique solution that satisfies NE, Eff, LCons, and CoLCons is the Bird rule.

Proof : The Bird rule has the properties, and if another solution has the properties, by lemma 2.2.6, it coincides with the Bird rule. □

The properties used to characterize the Bird rule are logically independent. We show this by giving examples of solutions that satisfy three of the four properties.

Example 2.2.8 If we leave out the non-emptiness property, the empty solution that assigns the empty set to every mcst problem satisfies Eff, LCons and CoLCons.

Example 2.2.9 If we leave out the efficiency property, the solution that assigns $((e)_{e\in E_{N^*}}, (a, \ldots, a))$ to every mcst problem, satisfies the other three properties. Here $(e)_{e\in E_{N^*}}$ denotes the sequence of all edges of the complete graph on N^* ordered by non-decreasing cost, and $a = \sum_{e\in E_{N^*}} w(e)$. Notice that there are no leaves in the complete graph, except if there is only one player; the leaf consistency property is thus trivially satisfied.

Example 2.2.10 If leaf consistency is left out, the solution that assigns to an mcst problem $\langle N, *, w\rangle$ the set of all efficient outcomes

$$\{((e^1, \ldots, e^\tau), x) \mid \langle N^*, \{e^1, \ldots, e^\tau\}\rangle \text{ is an mcst, } x \in \mathbf{R}^N \text{ and } x(N) = \sum_{t=1}^{\tau} w(e^t)\}$$

satisfies the three other properties.

For the last example, we will use a total ordering $<$ on the universe of all possible players. This is possible : usually, names of players are finite strings in some finite alphabet, that can be alphabetically ordered. Define the lexicographical order on elements of a solution to an mcst problem $\langle N, *, w\rangle$ by $((e^1, \ldots, e^\tau), x) \prec_L ((\tilde{e}^1, \ldots, \tilde{e}^{\tilde{\tau}}), y)$ if there exists a $k \in N$ such that $x_i = y_i$ for $i < k$ and $x_k < y_k$.

Example 2.2.11 If converse leaf consistency is left out, the solution that assigns to every mcst problem the set of lexicographically smallest elements of the Bird rule satisfies the three other properties, but does not coincide with the Bird rule on all mcst problems; it does not, thus, satisfy converse leaf consistency.

2.3 Sustaining the Bird rule by Nash equilibria

The previous sections studied mcst problems by means of cooperative games. This section analyzes the problems using associated strategic mcst games, suggested by Jose Zarzuelo. An action of a player in the strategic mcst game consists of a specification of the edge that this player will construct, if any.

Definition 2.3.1 To a minimum-cost spanning tree problem $\langle N, *, w\rangle$, Zarzuelo (private communication) associates the *strategic mcst game* $\langle N, (A^i)_{i\in N}, (u_i)_{i\in N}\rangle$ in normal form with player set N, and in which an action $a^i \in A^i = E_{N^*} \cup \{\emptyset\}$ specifies which edge (if any at all) player i is willing to construct. The utility that player i derives from a strategy profile $a = (a^i)_{i\in N}$ is determined in the following way. We assume that players dislike constructing edges, but they absolutely have to be connected to the source. The utility function is thus linear in the cost of the edge constructed (if any), and a big penalty is subtracted if the player is not connected to the source in the graph $\langle N^*, C_a\rangle$. Here, for a strategy profile a, the set $C_a = \{a^i \mid i \in N \text{ and } a^i \in E_{N^*}\}$ is the set of edges that have been constructed and that will be constructed.

Formally, the utility of player i is defined as

$$u_i(a^1,\ldots,a^n) := \begin{cases} -w(a^i) & \text{if } i \text{ is connected to the source in } \langle N^*, C_a\rangle \\ -w(a^i) - P & \text{otherwise} \end{cases}$$

where P is a large number ($P > \sum_{e \in E_{N^*}} w(e)$). For convenience, define $w(\emptyset) = 0$.

We will proceed to establish a relationship between the Bird rule presented in section 2.2 and the Nash equilibria of the above strategic mcst game.

Theorem 2.3.2 Each element $((e^1,\ldots,e^\tau),x)$ of the Bird rule of an mcst problem corresponds to a Nash equilibrium of the associated strategic mcst game, in which the strategy of a player i is to construct the first edge on the unique path from i to the source in the tree $\langle N^*, \{e^1,\ldots,e^\tau\}\rangle$ and in which his payoff equals $-x_i$.

Proof : Let $\mathcal{T} = \langle N, *, w\rangle$ be an mcst problem and let $((e^1,\ldots,e^\tau),x)$ be an element of the Bird rule $\beta(\mathcal{T})$. The corresponding strategy a^i for a player i is the first edge e_i that lies on the unique path from i to the source in the tree $\langle N^*, \{e^1,\ldots,e^\tau\}\rangle$. If every player plays this strategy, the resulting set C_a of constructed edges is precisely $\{e^1,\ldots,e^\tau\}$, which implies that all players are connected to the source. So the payoff to player i equals $-w(e_i)$. Hence $u_i(a) = -w(e_i) = -x_i$.

To prove that a is a Nash equilibrium, suppose that a player i deviates from a. Now player i wants to avoid the penalty, which is larger than $w(e_i)$, so if i does not choose $a^i = e_i$, then it has to choose another edge e' that connects the component of i in the graph $\langle N^*, C_a \setminus \{e_i\}\rangle$ to the component of the source. Because $\langle N^*, \{e^1,\ldots,e^\tau\}\rangle$ is an mcst, such an edge e' has to be at least as costly as the edge e_i. Hence i is not better off. □

If the costs of all edges are positive, it can be proved that every Nash equilibrium of the strategic game, together with its payoff vector, corresponds to an element of the Bird rule.

Theorem 2.3.3 In an mcst problem in which the costs of every edge are positive, each Nash equilibrium a of the strategic mcst game specifies a minimum-cost spanning tree $\langle N^*, C_a\rangle$ for the mcst problem, and the payoff vector equals $-\beta^{C_a}$.

Proof : Let $\langle N, *, w\rangle$ be an mcst problem in which the costs of all edges is positive and let $\langle N, (A^i)_{i\in N}, (u_i)_{i\in N}\rangle$ be the associated strategic mcst game. Let $a = (a^1,\ldots,a^n)$ be a Nash equilibrium and consider the set C_a of edges that have been constructed. If a player i is not connected to the source in the graph $\langle N^*, C_a\rangle$, then by deviating and using the strategy $\hat{a}_i = \{i, *\}$, player i is connected to the source, thereby avoiding the penalty and improving his payoff. So in a Nash equilibrium, every player is connected to the source. Furthermore, if a cycle were present in the graph $\langle N^*, C_a\rangle$, some player is not connected to the source : there are at most $|N|$ edges constructed.

Hence, these strategies do not constitute a Nash equilibrium. $\langle N^*, C_a \rangle$ is thus a spanning tree.

This means that $|N|$ edges are constructed; every player thus constructs an edge. Furthermore, because all edges have a positive cost, every player i pays an edge only if it lies on the path in the tree from i to the source. If this were not true, some player could deviate by choosing to construct no edge, and would thus not incur the penalty. Hence, this player would profit from this deviation.

By induction on the number of players on the path between a player and the source, one deduces that every player i pays the edge incident to i, on the path from i to the source in the tree $\langle N^*, C_a \rangle$. Hence, the payoff vector equals $-\beta^{C_a}$.

To prove that $\langle N^*, C_a \rangle$ is a minimum-cost spanning tree, compare it with the mcst $\langle N^*, F \rangle$ that is constructed using Prim and Dijkstra's algorithm (see section 2.1), by choosing, whenever possible, an edge in C_a. Look at the first edge $e \in F$ chosen by Prim and Dijkstra's algorithm, which does not lie in C_a. If this edge exists, it connects a player i with the component of the source and is *strictly* cheaper than a_i, the edge that this player i constructs. (If the edge e had the same cost as a_i, it would have been chosen by Prim-Dijkstra's algorithm.) If i constructs e instead of a_i, player i would still be connected to the source, and thus pay less. This contradicts the fact that a is a Nash equilibrium, hence the edge e cannot be found. This implies that $F = C_a$, which means that $\langle N^*, C_a \rangle$ is a mcst. □

If there is an edge that costs nothing, the graph constructed in a Nash equilibrium is still a mcst, but the costs do not have to be divided according to Bird's tree-allocation rule.

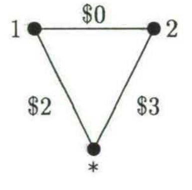


Figure 2.2: a Nash equilibrium unrelated to Bird's tree allocations.

Example 2.3.4 Consider the problem drawn in figure 2.2. The strategy pair in which player 1 pays the edge $\{1, 2\}$ and player 2 pays $\{1, *\}$ is a Nash equilibrium in the associated strategic game, but the associated payoff vector $(0, -2)$ does not correspond to Bird's unique tree allocation $(2, 0)$.

Chapter 3

Minimum-cost spanning extension problems

This chapter, which is based on Feltkamp, Tijs and Muto (1994b), generalizes minimum-cost spanning tree problems to minimum-cost spanning extension problems. These are network construction problems in which some network can be present initially, which has to be extended to a network connecting every player to the source.

Mathematically, this generalization has the advantage that a half-solved mcst or mcse problem is again an mcse problem, which allows a recursive solution and from an applied point of view the advantage is that more problems can be treated. If the original problem was suggested by electrification of Moravia at the beginning of the century, the problem by now is how to extend a network already present and allocate the cost of the extension.

As before, finding a minimum-cost spanning extension is only part of the problem : if the villages must bear the cost of this extension, a cost-allocation problem has to be addressed as well.

As in chapter 2, we integrate the problem of constructing an optimal network and the problem of allocating the costs. Whereas in chapter 2 Prim-Dijkstra's algorithm for constructing an mcst appeared to be closely linked to Bird's tree allocations, here we show that the irreducible core and Kruskal's (1956) algorithm for constructing an mcst are related and also provide a one-point refinement, the equal-remaining-obligations solution. Moreover, we provide axiomatic characterizations of the irreducible core and the equal-remaining-obligations solution.

The outline of this chapter is as follows.

Section 3.1 presents a formal model of the *minimum-cost spanning extension* (*mcse*) problem in which an existing network has to be extended to a minimum-cost spanning network, i.e. a network that connects every village to the source and that offers minimum cost among all such networks. An algorithm to find a minimum-cost spanning extension and an associated set of cost allocations is presented. This algorithm is similar to Kruskal's (1956) algorithm. We prove that the set of allocations generated is a subset of the core of the associated mcse game and that it is independent of the extension that is constructed.

Section 3.2 generalizes the definition of the irreducible core, as proposed in Bird (1976) for minimum-cost spanning tree problems, to minimum-cost spanning extension problems and proves that the set of allocations generated by the algorithm in section 3.1 coincides with the irreducible core. A corollary is that Bird's tree allocations (see Bird (1976)) for minimum-cost spanning tree problems are also generated by our algorithm.

Section 3.3 introduces the equal-remaining-obligations (ERO) rule, a one-point refinement of the irreducible core. It is obtained by an algorithm similar to the algorithm for the irreducible core presented in section 3.1. Like the irreducible core, the ERO rule is independent of the extension constructed. This contrasts with Bird's tree-allocation rule, which depends on the tree constructed.

Section 3.4 axiomatically characterizes the irreducible core and the equal-remaining-obligations rule. Efficiency, consistency and converse consistency are among the axioms we use.

The rather lengthy proofs of the main theorems of section 3.1 are provided in an appendix, section 3.5.

3.1 Mcse problems : a solution

This section formally presents minimum-cost spanning extension problems, mcse games and an algorithm that computes for any mcse problem a minimum-cost spanning extension and an associated set of allocations, which turns out to be contained in the core of the mcse game.

A *minimum-cost spanning extension problem* consists of a set N of users who have to extend an existing network in order to be connected to a source, denoted by $*$. The links are costly and the users have to pay for the extension. Such a problem is represented by a complete graph $\langle N^*, E_{N^*}\rangle$ on the set N^* containing all users and the source, together with a set $E \subseteq E_{N^*}$ of already constructed links and a weight function $w : E_{N^*} \to \mathbf{R}_+$. The cost of constructing an edge e is given by the positive weight $w(e) > 0$ of this edge. The edges in E can be costlessly used. Because the graph of possible edges is always the complete graph, we denote an mcse problem with set of users N, source $*$, weight function w and existing edge set E by $\langle N, *, w, E\rangle$. If the set of existing edges E is empty, the mcse problem becomes the classical minimum-cost spanning tree problem and instead of writing $\langle N, *, w, \emptyset\rangle$, we will write $\langle N, *, w\rangle$.

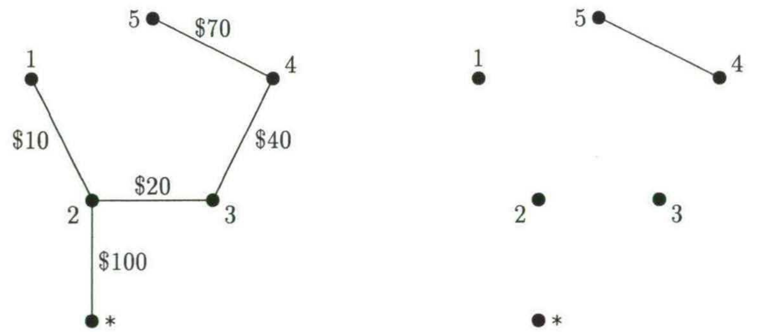
Mcse problems can be split up into two subproblems, an Operations Research problem of connecting all users to the source by means of an extended graph $\langle N^*, E \cup E'\rangle$ such that the cost of the extension E' is minimal, and a cost-allocation problem of allocating this cost to the users in a reasonable way.

In the special case of mcst problems, an mcst can be constructed by Kruskal's algorithm.

Algorithm 3.1.1 (Kruskal 1956)
input : an mcst problem $\langle N, *, w\rangle$
output : an edge set E' of a minimum-cost spanning tree

1. start with the empty set $E' = \emptyset$.
2. Find an edge $e \notin E'$ of minimum cost such that the graph $\langle N^*, E' \cup \{e\}\rangle$ does not contain a cycle.
3. Join this edge to the set E' $(E' := E' \cup \{e\})$.
4. If the graph $\langle N^*, E'\rangle$ is not connected, go back to step 2.
5. E' is the required edge set.

For mcse problems, a generalization of Kruskal's algorithm is demonstrated in example 3.1.2.



The weights of links. $\{4, 5\}$ is already constructed.

Figure 3.1: A simple mcse problem

Example 3.1.2 Let $N = \{1, 2, 3, 4, 5\}$ and let the weights of the edges and the graph which is already constructed be as in figure 3.1. The costs of edges that are not indicated are \$200. First construct the edge $\{1, 2\}$; it is the cheapest one that introduces no cycle. The same reasoning picks $\{2, 3\}$ as second edge, as third edge $\{3, 4\}$ and finally as last edge, $\{2, *\}$.

The algorithm demonstrated is the following :

Algorithm 3.1.3 (Kruskal generalized to mcse problems)
input : an mcse problem $\mathcal{M}$
output : an mcse

1. Given $\mathcal{M} \equiv \langle N, *, w, E\rangle$, define

$$\begin{array}{rcll} t & = & 0 & \text{the initial stage,} \\ \tau & = & |N^*/E| - 1 & \text{the number of stages,} \\ E^0 & = & \emptyset & \text{the initial edge set.} \end{array}$$

2. While $t < \tau$, do steps 3 to 5.

3. $t := t + 1$.

4. At stage t, given E^{t-1}, choose a cheapest edge $e^t \notin E \cup E^{t-1}$ such that the graph $\langle N^*, E \cup E^{t-1} \cup \{e^t\}\rangle$ contains no more cycles than the graph $\langle N^*, E \cup E^{t-1}\rangle$.

5. Define $E^t := E^{t-1} \cup \{e^t\}$.

6. E^τ is the extension we were seeking. Denote the sequence of edges constructed by $\mathcal{E} = (e^1, \ldots, e^\tau)$.

Lemma 3.5.1 proves that the extension constructed is indeed an mcse. Note the algorithm constructs the edges of a minimum-cost spanning extension in the order of non-decreasing costs. It is easy to see that any sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ that consists of the edges of a minimum-cost spanning extension ordered by non-decreasing costs can be constructed with algorithm 3.1.3 by a suitable choice of the edges chosen in step 4.

Bird (1976) associates a tree allocation with every minimum-cost spanning tree in an mcst problem. Chapter 2 proved that this tree allocation can be associated with the sequence of edges that Prim and Dijkstra's algorithm generates when generating this mcst. This suggests looking for allocations associated with the sequences generated by Kruskal's algorithm.

For an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$, the minimum-cost spanning extensions E' have one edge less than the number of components $|N^*/E|$ (if more edges were built, a cycle would be introduced, which cannot be minimal in cost, as the weights of the edges are positive). Hence, defining $\tau := |N^*/E| - 1$, associate an allocation with every sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ of edges that introduces no new cycle in the mcse problem $\mathcal{M}$. Note that any such sequence connects all players to the source. The idea behind the allocation is that at each successive stage t, the cost of the edge e^t that is constructed at stage t is shared among the players in N according to a *fraction vector* $f^t \in \Delta^N$. Three rules have to be observed when allocating the cost of e^t:

- At stage t, edge e^t connects two components of the graph $\langle N^*, E \cup \{e^1, \ldots, e^{t-1}\}\rangle$, creating a component C^t of the graph $\langle N^*, E \cup \{e^1, \ldots, e^t\}\rangle$. Only players in C^t contribute to the cost of e^t.

- The players in the component C_*^{t-1} of the source in the graph $\langle N^*, E \cup \{e^1, \ldots, e^{t-1}\}\rangle$ constructed before stage t do not contribute to the cost of e^t.

- Furthermore, summing over all edges in the sequence, every component of the original graph $\langle N^*, E\rangle$ that does not contain the source pays fractions of edges to a total of one.

Hence, define the set $V^{\mathcal{E}}(\mathcal{M})$ of sequences of fraction vectors *valid* for the sequence $\mathcal{E}$ in $\mathcal{M}$ by

$$V^{\mathcal{E}}(\mathcal{M}) := \left\{ (f^1, \ldots, f^\tau) \in \mathbf{R}^{\tau|N|} \;\middle|\; \begin{array}{rcl} \sum_{k \in C^t} f_k^t & = & 1 \quad \text{for all } t \\ & \text{and} & \\ \sum_{k \in C} \sum_{s=1}^{\tau} f_k^s & = & 1 \quad \forall C \in N^*/E \text{ with } * \notin C \\ & \text{and} & \\ f_k^t & = & 0 \quad \text{for all } t, k \text{ s.t. } k \in C_*^{t-1} \end{array} \right\}.$$

For a sequence $\mathcal{F} \equiv (f^1, \ldots, f^\tau)$ valid for $\mathcal{E}$ in $\mathcal{M}$, define the allocation

$$x^{\mathcal{E},\mathcal{F}}(\mathcal{M}) := \sum_{t=1}^{\tau} f^t w(e^t) \in \mathbf{R}^N \tag{3.1.1}$$

and define the set $D^{\mathcal{E}}(\mathcal{M})$ by

$$D^{\mathcal{E}}(\mathcal{M}) := \{x^{\mathcal{E},\mathcal{F}} \mid \mathcal{F} \in V^{\mathcal{E}}\}. \tag{3.1.2}$$

If no confusion can occur, we drop the argument $\mathcal{M}$.

Lemma 3.1.4 For all mcse problems $\mathcal{M}$, for all sequences $\mathcal{E} = (e^1, \ldots, e^\tau)$ such that $E^\tau := \{e^1, \ldots, e^\tau\}$ is an mcse of $\mathcal{M}$ and all $\mathcal{F}$ valid for $\mathcal{E}$, the allocation $x := x^{\mathcal{E},\mathcal{F}}(\mathcal{M})$ is efficient : $\sum_{i \in N} x_i = c^{\mathcal{M}}(N)$.

Proof : Validity of $\mathcal{F}$ implies every edge $e^t \in E^\tau$ is paid for by the component C^t it constructs. Secondly, E^τ is a minimal-cost spanning extension of $\langle N^*, E\rangle$. Hence,

$$\sum_{i \in N} x_i = \sum_{e \in E^\tau} w(e) = c^{\mathcal{M}}(N).$$

□

Note that because the set of valid sequences of fraction vectors $V^{\mathcal{E}}$ is convex and the map

$$x^{\mathcal{E}} : \mathcal{F} \mapsto x^{\mathcal{E},\mathcal{F}}$$

is linear, the set $D^{\mathcal{E}}$ is also convex, for any sequence $\mathcal{E}$.

Instead of first constructing the edges and later allocating their cost, one could allocate the cost of the edge e^t immediately, because the validity of a sequence of fraction vectors can be checked stage by stage : a sequence $f^1, \ldots, f^\tau$ is valid for $e^1, \ldots, e^\tau$ in $\mathcal{M}$ if and only if at every stage t it satisfies

- the component C^t constructed at stage t pays the cost of the edge e^t,
- for every component in the original graph $\langle N^*, E\rangle$, the total of the fractions paid up to stage t does not exceed 1,
- the people outside C^t pay nothing,
- the people in the component C_*^{t-1} of the source pay nothing.

In formula, this gives

$$\begin{cases} \sum_{k\in C^t} f_k^t &= 1, \\ \sum_{k\in C}\sum_{s=1}^{t} f_k^s &\leq 1 \quad \text{for all } C \in N^*/E, \\ f_k^t &= 0 \quad \text{if } k \notin C^t, \\ f_k^t &= 0 \quad \text{if } k \in C_*^{t-1} \end{cases} \tag{3.1.3}$$

for every stage t.

Example 3.1.5 For an mcst problem $\mathcal{T} \equiv \langle N, *, w\rangle$, Prim and Dijkstra's algorithm (see chapter 2) constructs a sequence $\mathcal{E} = (e^1, \ldots, e^{|N|})$ of edges leading to an mcst $\langle N^*, T\rangle$ as follows : at every stage t, e^t is an edge that connects a player with the component of the source in the graph $\langle N^*, \{e^1, \ldots, e^{t-1}\}\rangle$ and which has minimal cost among all such edges. Without loss of generality, we number the players in N such that for every t, the edge e^t connects player t with the component of the source, $\{*, 1, \ldots, t-1\}$. Hence, the edge e^1 connects player 1 to the source, and under the system (3.1.3), player 1 has to pay the cost of e^1 and the other players do not contribute. In the second stage, player 2 is connected to the component of the source, which now equals $\{*, 1\}$. The first equation in system (3.1.3) implies players 1 and 2 pay the cost of edge e^2. The fourth implies that player 1, who is in the component of the source, does not contribute, hence, player 2 is assigned the cost of edge e^2. The third equation implies the other players do not contribute. The inequality is satisfied, because up to now, every component in the original graph (i.e. every player) paid either one edge or no edges. By induction, we see that at every stage, the component C^t consists of the component C_*^{t-1} and the newly connected player t. Because the component of the source does not contribute to the cost of e^t, the unique valid allocation of the cost of this edge is to allocate it completely to player t. Hence, $D^{\mathcal{E}}(\mathcal{T})$ consists of one allocation, in which each player i is allocated the cost of the edge incident to i on the unique path in the tree from i to the source. This allocation is precisely Bird's tree allocation β^T associated with the mcst $\langle N^*, T\rangle$.

Example 3.1.6 Computing the extreme points of the set of valid fraction vectors for the sequence $\mathcal{E}$ constructed in example 3.1.2 shows that in this case $D^{\mathcal{E}}(\mathcal{M})$ is the convex hull of the vectors

$$\begin{array}{llll}
(10,20,40,100,0), & (10,20,40,0,100), & (10,20,100,40,0), & (10,20,100,0,40),\\
(10,40,20,100,0), & (10,40,20,0,100), & (10,100,20,40,0), & (10,100,20,0,40),\\
(20,10,40,100,0), & (20,10,40,0,100), & (20,10,100,40,0), & (20,10,100,0,40),\\
(40,10,20,100,0), & (40,10,20,0,100), & (100,10,20,40,0), & (100,10,20,0,40).
\end{array}$$

Inspired by Bird's mcst game (see chapter 2), we associate a minimum-cost spanning extension game $(N, c^{\mathcal{M}})$ with an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ as follows. Each coalition $S \subseteq N$, if it cannot count on the players in its complement, has to solve a problem similar to the problem of the grand coalition, namely, extending the existing graph to a graph connecting all users in S to the source. The cost of this extension is the worth $c^{\mathcal{M}}(S)$ of coalition S in the mcse game.

When computing the cost of a coalition S, several questions arise. Can the coalition use all or some of the edges that are already present? Is it allowed to use vertices outside S? We opt for the following answers : a coalition S is allowed to use all edges that are initially present, but can only use those vertices that lie in a component of $\langle N^*, E\rangle$ that contains members of S or the source. Now, consider an example to clarify all this.

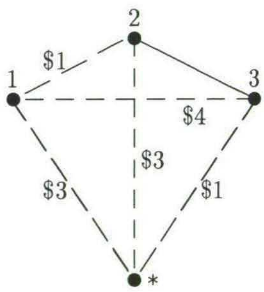


Figure 3.2: $\{1,2\}$ is allowed to use the edge $\{2,3\}$, but $\{1\}$ is not.

Example 3.1.7 In the problem depicted in figure 3.2, edge $\{2,3\}$ is already constructed. Coalition $\{1,2\}$ is allowed to use the edge $\{2,3\}$ and can connect itself by building the edges $\{1,2\}$ and $\{3,*\}$; so, $c(\{1,2\}) = 1 + 1 = 2$. Coalition $\{1\}$ is not allowed to use the edge $\{2,3\}$ because the component $\{2,3\}$ has no vertices in common with $\{1\}$ or the source, hence $c(\{1\}) = 3$. The other worths are $c(\{2\}) = c(\{3\}) = 1$ (connect player 2 via player 3), and finally $c(\{1,3\}) = c(N) = 2$.

In general, the formula becomes

$$c^{\mathcal{M}}(S) := \min\left\{ \sum_{e\in E'} w(e) \mid \begin{array}{l} S \subset C_*^{E'} \text{ and } E' \text{ contains only edges between} \\ \text{components of } N^*/E \text{ containing members of } S^* \end{array} \right\}$$

for all $S \subseteq N$, where $C_*^{E'}$ is the component of the source $*$ in the graph $\langle N^*, E \cup E' \rangle$.

The next theorem states that an allocation which is associated with a sequence of edges generated by algorithm 3.1.3 is a core element of the mcse game.

Theorem 3.1.8 For any mcse problem $\mathcal{M}$, for any sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 3.1.3 applied to $\mathcal{M}$ and any sequence of fraction vectors $\mathcal{F}$ valid for $\mathcal{E}$ the allocation $x^{\mathcal{E},\mathcal{F}}$, as defined in equation (3.1.1), is a core-allocation of the mcse game $(N, c^{\mathcal{M}})$ associated with $\mathcal{M}$.

The proof of this theorem is lengthy and technical, and can be found in the appendix. An immediate consequence is

Corollary 3.1.9 For any sequence $\mathcal{E}$ leading to a minimum-cost spanning extension for an mcse problem $\mathcal{M}$ with associated mcse game $(N, c^{\mathcal{M}})$,

$$D^{\mathcal{E}}(\mathcal{M}) \subseteq \mathrm{Core}(N, c^{\mathcal{M}}).$$

Another way to generate mcse games is to define the cost of a coalition S as the minimal cost of an extension connecting S to the source, without any restriction on the vertices of the extension. This approach yields a monotonic game, with the same cost for the grand coalition, but smaller costs for the other coalitions, because these now have more opportunities to save costs. Hence, the core of this variant is in general contained in the core of the mcse game already defined. However, such a monotonic game associated with an mcse problem $\langle N, *, w, E \rangle$ can be considered as an mcse game according to our definition associated to the mcse problem $\langle N, *, w', E \rangle$, in which the weights of links have been (iteratively) reduced to satisfy the triangle inequality

$$w'(\{i,j\}) \leq w'(\{i,k\}) + w'(\{k,j\}) \qquad \text{for all } i,j,k \in N^*.$$

Thus, this introduces no new games. Moreover, core elements of the monotonic game can be computed by applying algorithm 3.1.3 to the problem with reduced weights.

In computing the cost of a coalition in an mcse game, we allow this coalition to use the vertices in its complement to which it is connected via edges in E. A second possible variation is *not* to allow a coalition to use any players in its complement when connecting to the source. This would yield a game with the same cost for the grand coalition, but larger costs for the other coalitions, as these are now restricted in their possibilities. Hence, the core of this variant contains the core of our mcse game, which implies all algorithms presented in this paper yield core elements of the variant.

A question that arises is the following. How does the set $D^{\mathcal{E}}$ depend on the sequence $\mathcal{E}$? It is answered in the next proposition.

Proposition 3.1.10 For any mcse problem $\mathcal{M}$, for all sequences $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ constructed by the algorithm 3.1.3 applied to $\mathcal{M}$,

$$D^{\mathcal{E}}(\mathcal{M}) = D^{\widetilde{\mathcal{E}}}(\mathcal{M}).$$

A proof can be found in the appendix.

Because $D^{\mathcal{E}}$ is independent of the sequence $\mathcal{E}$ of edges, as long as this sequence is constructed by the algorithm 3.1.3, we define for an mcse problem $\mathcal{M}$:

$$D^{GK}(\mathcal{M}) := D^{\mathcal{E}}(\mathcal{M})$$

for any sequence $\mathcal{E}$ obtained by the algorithm 3.1.3 applied to $\mathcal{M}$. (The superscript GK stands for generalized Kruskal).

3.2 The irreducible core of mcse problems

This section will generalize the irreducible core from mcst problems to mcse problems and prove that the set D^{GK} coincides with this irreducible core.

Definition 3.2.1 Given an mcse problem $\langle N, *, w, E\rangle$, we define the associated mcst problem $\langle N_E, *_E, w_E\rangle$ as follows : N_E consists of the components of $\langle N^*, E\rangle$ that do not contain the source $*$; the new source $*_E$ is the component of $\langle N^*, E\rangle$ which contains the original source $*$, and w_E is defined by

$$w_E(C, D) := \min\{w(i,j) \mid i \in C, j \in D\}$$

for all components C and D of the graph $\langle N^*, E\rangle$. Furthermore, for an edge $e = \{i,j\} \in E_{N^*}$, define

$$e_E := \{C_i, C_j\} \tag{3.2.1}$$

and for a set of edges $F \subseteq E_{N^*}$, define

$$F_E := \{\{C_i, C_j\} \mid \{i,j\} \in F\}, \tag{3.2.2}$$

where C_i and C_j are the components of $\langle N^*, E\rangle$ containing players i and j, respectively.

The intuitive idea is to contract each component not containing the source into a single player and to contract the component of the source into a new source. Note that if i and j lie in the same component of the graph $\langle N^*, E\rangle$, then the edge $\{i,j\}/E$ has two identical end points, i.e. it is a loop.

It is easy to see that if F is an mcse of the mcse problem $\langle N, *, w, E\rangle$, then the tree $\langle N_E^*, F_E\rangle$ is an mcst of the associated mcst problem $\langle N_E, *_E, w_E\rangle$. Conversely, if $\langle N_E^*, T\rangle$ is an mcst of the associated mcst problem, then there exists an mcse F with $F_E = T$. This correspondence, though possibly associating several mcses with one mcst, transfers the well-known structure of the collection of mcs trees of an mcst problem onto the set of mcs extensions of an mcse problem.

Example 3.2.2 Consider the mcse problem $\langle N, *, w, E\rangle$ depicted in figure 3.3, in which all edges cost \$1 and edge $\{1,2\}$ is already constructed. There are two mcses in this mcse problem. In the associated mcst problem, there is only one player, so the unique mcst consists of the edge connecting this player to the source. In the mcse problem, both the edge $\{*,1\}$ and the edge $\{*,2\}$ correspond to this edge of the mcst problem.

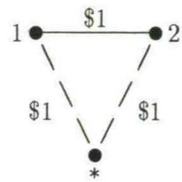


Figure 3.3: A simple mcse problem with two mcses.

More about the structure of the set of mcses will be said in the appendix, in the proof of proposition 3.1.10.

We now define the irreducible core of an mcse problem. It is a straightforward generalization of the definition of the irreducible core of an mcst game provided in Bird (1976). Apparently, it depends on an mcse. However, this is not the case, as will be proved later.

Definition 3.2.3 Given an mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$ and an mcse E', define the *irreducible core* $\mathrm{IC}(\mathcal{M}, E')$ of $\mathcal{M}$ with respect to E' as follows : consider the set $\mathrm{Var}(\mathcal{M}, E')$ of all mcse problems that have E' as mcse and that are obtained from $\mathcal{M}$ by varying the weights $w(e)$ of edges $e \notin E'$ which connect two components of $\langle N^*, E\rangle$. Now $\mathrm{IC}(\mathcal{M}, E')$ is the intersection of the cores of all mcse games associated with an mcse problem in $\mathrm{Var}(\mathcal{M}, E')$, i.e.

$$\mathrm{IC}(\mathcal{M}, E') := \bigcap \left\{ \mathrm{Core}(N, c^{\mathcal{M}'}) \mid \mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E') \right\}.$$

If the set E of initially present edges is empty, the present definition coincides with the definition of irreducible core of an mcst problem in Bird (1976). For mcst problems, it is already known that the irreducible core is independent of the mcst used to define it.

Equivalently, the irreducible core could be defined as follows : given an mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$ and an mcse E', for any two players $i, j \in N$, let P_{ij} be a path in the graph $\langle N^*, E \cup E'\rangle$ from i to j. It is possible that this path is not unique, but the part of the path in E' is. Define a new cost function

$$\underline{w}(i,j) := \begin{cases} \max\{w(e') \mid e' \in P_{ij} \cap E'\} & \text{if } C_i \neq C_j, \\ w(\{i,j\}) & \text{if } C_i = C_j, \end{cases} \tag{3.2.3}$$

where C_i and C_j are the components of $\langle N^*, E\rangle$ containing i and j, respectively. Note that if $\{i,j\} \in E'$ then P_{ij} consists only of the edge $\{i,j\}$, hence $\underline{w}(\{i,j\}) = w(\{i,j\})$.

Then the following holds :

Theorem 3.2.4 The irreducible core $\mathrm{IC}(\mathcal{M}, E')$ of an mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$ coincides with the core of the game $(N, \underline{c}) = (N, c^{\langle N, *, \underline{w}, E\rangle})$.

Proof : First, E' is an mcse of the problem $\langle N, *, \underline{w}, E\rangle$. This can be seen as follows. Suppose there is a spanning extension $\tilde{E}$, which is cheaper than E'. Order the edges in $\tilde{E}$ by increasing weight and take the first edge $\{i,j\} = e \in \tilde{E} \setminus E'$. Adding e to $\langle N^*, E \cup E' \rangle$ creates a cycle formed by e and the edges in P_{ij}. Hence, there exists an edge $e' \in P_{ij} \cap E' \setminus \tilde{E}$, such that substituting e with e' in $\tilde{E}$ yields another spanning extension of the problem $\langle N, *, \underline{w}, E \rangle$. Now e connects two components of $\langle N^*, E \rangle$; hence $\underline{w}(e) \geq w(e')$ for all edges $e' \in P_{ij} \cap E'$. This spanning extension is thus an mcse, which, moreover, has one more edge in common with E'. Repeat this argument enough times to see that E' is indeed an mcse of the problem $\langle N, *, \underline{w}, E \rangle$, which implies that $\langle N, *, \underline{w}, E \rangle \in \text{Var}(\mathcal{M}, E')$. Hence, the irreducible core of $\mathcal{M}$ is included in the core of the game $(N, \underline{c})$.

On the other hand, note that for an edge $e = \{i,j\} \notin E'$ connecting two components C_i and C_j, $\max\{w(e') \mid e' \in P_{ij} \cap E'\}$ is a lower bound on the weight that can be assigned to this edge e in an mcse problem $\mathcal{M}' \in \text{Var}(\mathcal{M}, E')$: if a lower weight is assigned to e, then there is an edge $e' \in P_e \cap E'$ with higher weight than e and replacing e' by e would yield a spanning extension $E' \setminus \{e'\} \cup \{e\}$ which has lower cost than E'. Hence, $\underline{w}(e) \leq w'(e)$ for all edges $e \in E_{N^*}$, for every problem $\langle N, *, w', E \rangle \in \text{Var}(\mathcal{M}, E')$. This implies that $\underline{c}(S) \leq c^{\mathcal{M}'}(S)$ for all coalitions S and for all mcse problems $\mathcal{M}'$ in $\text{Var}(\mathcal{M}, E')$. Because E' is an mcse of $\langle N, *, \underline{w}, E \rangle$, it also follows that $\underline{c}(N) = \sum_{e \in E'} w(e) = c^{\mathcal{M}'}(N)$. Hence, $\text{Core}(\underline{c}) \subseteq \text{Core}(c^{\mathcal{M}'})$ for all $\mathcal{M}' \in \text{Var}(\mathcal{M}, E)$, which implies that the core of $\underline{c}$ is included in the irreducible core of $\mathcal{M}$. Together with the first part of the proof, this proves the theorem. □

Example 3.2.5 Computing the reduced weights in the mcse problem of example 3.1.2, we obtain $\underline{w}(\{1,5\}) = \underline{w}(\{1,4\}) = \$40$, $\underline{w}(\{1,3\}) = \$20$, $\underline{w}(\{1,2\}) = \$10$, $\underline{w}(\{2,5\}) = \underline{w}(\{2,4\}) = \$40$, $\underline{w}(\{2,3\}) = \$20$, $\underline{w}(\{3,5\}) = \underline{w}(\{3,4\}) = \$40$, $\underline{w}(\{4,5\}) = \$70$ and $\underline{w}(\{i,*\}) = \$100$ for each player i.

In order to analyze the structure of the irreducible core of an mcse problem, we relate its structure with the structure of the irreducible core of the associated mcst problem of definition 3.2.1, using the concept of marionettes. Zumsteg (1992) defined two players i, j in a game (N, c) to be marionettes if

$$c(S \cup \{i\}) = c(S \cup \{j\}) = c(S \cup \{i,j\})$$

for all $S \subseteq N$. Considering players to be marionettes of themselves turns being marionettes into an equivalence relation; we denote it by $\sim$. For any player i, the set of marionettes of i is denoted by S_i.

Definition 3.2.6 For a game (N, c), the marionette-reduced game (N', c') is the game in which $N' = \{S_i \mid i \in N\}$ and which satisfies $c'(C) = c(\bigcup_{S \in C} S)$ for all $C \in N'$. Hence a player in the marionette-reduced game consists of all marionettes of one player in the original game.

Equivalently, one could obtain the marionette-reduced game as a subgame of the original game : for each player $U \in N'$, take one representative player $j_U \in U$ and define $T = \{j_U \mid U \in N'\}$. Define the subgame (T, c^T) by

$$c^T(U) = c(U)$$

for all $U \subseteq T$. For every player i in T, there exists a unique player S_i in N' satisfying $j_{S_i} = i$ and for every player U in N', there exists a unique player j_U in T satisfying $S_{j_U} = U$. Furthermore this bijection between the players of T and N' turns out to be an isomorphism between the games (N', c') and (T, c^T) :

$$\begin{array}{rclclcl} c'(C) & = & c(\bigcup_{S \in C} S) & = & c(\{j_S \mid S \in C\}) & = & c^T(\{j_S \mid S \in C\}) \\ c^T(U) & = & c(U) & = & c(\bigcup_{i \in U} S_i) & = & c'(\{S_i \mid i \in U\}) \end{array}$$

for all coalitions $C \subseteq N'$ and $U \subseteq T$.

Lemma 3.2.7 If (N, c) is a game and (N', c') is its marionette-reduced game, their cores are related as follows :

1. if $x \in \text{Core}(N, c)$, then $y \in \text{Core}(N', c')$, where $y \in \mathbf{R}^{N'}$ is defined by $y_S = \sum_{i \in S} x_i$ for all $S \in N'$.
2. if $y \in \text{Core}(N', c') \cap \mathbf{R}_+^{N'}$, then $x \in \text{Core}(N, c)$, for all $x \in \mathbf{R}_+^N$ satisfying $y_S = \sum_{i \in S} x_i$ for all $S \in N'$. Moreover, such an x exists.

Proof : The proof of part 1 is trivial. To prove part 2, take $y \in \text{Core}(N', c')$. We first prove that an x that satisfies the requirements exists. For all $S \in N'$, choose a representative player $i_S \in S$ and assign

$$x_i := \begin{cases} y_S & \text{if } i = i_S \text{ for an } S \in N', \\ 0 & \text{otherwise.} \end{cases}$$

Because y is non-negative, so is x. Now $x(N) = x(\{i_S \mid S \in N'\}) = y(N') = c'(N') = c(N)$ and for any coalition $T \subseteq N$, there exists a subset T' of T, such that $\bigcup_{i \in T} S_i = \bigcup_{i \in T'} S_i$, where the right-hand side is a disjoint union. Hence,

$$\begin{array}{rclclcl} c(T) & = & c(\bigcup_{i \in T} S_i) & = & c(\bigcup_{i \in T'} S_i) & = & c'(\{S_i \mid i \in T'\}) \\ & \geq & \sum_{i \in T'} y_{S_i} & = & \sum_{i \in T'} x(S_i) & = & x(\bigcup_{i \in T'} S_i) \\ & \geq & x(T), & & & & \end{array}$$

which implies x satisfies the requirements.

Now take any $x \in \mathbf{R}_+^N$ satisfying $y_S = \sum_{i \in S} x_i$ for all $S \in N'$. Then

$$x(N) = \sum_{S \in N'} x(S) = \sum_{S \in N'} y_S = y(N') = c'(N') = c(N)$$

and for any coalition T, take again the subset T' such that $\bigcup_{i\in T} S_i = \bigcup_{i\in T'} S_i$, where the right-hand side is a disjoint union. Then because x is non-negative and $y \in \mathrm{Core}(N', c')$,

$$\begin{aligned} x(T) &\leq \sum_{i\in T'}\sum_{j\in S_i} x_j = \sum_{i\in T'} x(S_i) = \sum_{i\in T'} y_{S_i} \\ &\leq c'(\{S_i \mid i \in T'\}) = c(T') \\ &= c(T) \end{aligned}$$

Hence, $x \in \mathrm{Core}(N, c)$. □

Theorem 3.2.8 For a given mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$ with associated minimum-cost spanning tree problem $\mathcal{T} = \langle N_E, *_E, w_E\rangle$, the mcst game $(N_E, c^{\mathcal{T}})$ associated with $\mathcal{T}$ coincides with the marionette-reduction (N', c') of the mcse game $(N, c^{\mathcal{M}})$.

Proof : We first prove that the sets N_E and N' coincide. Two players that are in the same component of $\langle N^*, E\rangle$ are marionettes in the mcse game : if either one is connected to the source, so is the other, so that the cost of connecting one is the cost of connecting both. Hence, a player in N_E, being a component of $\langle N^*, E\rangle$, is a coalition of marionettes of the mcse game.

On the other hand, if two players are marionettes in the mcse game, it means that connecting one of them to the source is as costly as connecting the other player or connecting both players. Because the cost of all edges is positive, it follows that both players must lie in the same component.

Hence the set of players N_E in the mcst game coincides with the set of players N' in the marionette-reduction of the mcse game.

Consider a coalition $C \subseteq N'$. By definition of the associated mcst problem,

$$c^{\mathcal{T}}(C) = c^{\mathcal{M}}\Big(\bigcup_{S\in C} S\Big).$$

As for all $S \in C$, all players in S are marionettes in the mcse game, it follows that

$$c^{\mathcal{M}}\Big(\bigcup_{S\in C} S\Big) = c'(C).$$

Hence $c^{\mathcal{T}}(C) = c'(C)$, which concludes the proof. □

The next proposition states the relation between the irreducible core of an mcse problem and the associated mcst problem.

Proposition 3.2.9 For an mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$ and an mcse E', the irreducible core $\mathrm{IC}(\mathcal{M}, E')$ satisfies

$$\mathrm{IC}(\mathcal{M}, E') = \{x \in \mathbf{R}^N_+ \mid y \in \mathrm{IC}(\langle N_E, *_E, w_E\rangle) \text{ where } y_S := \sum_{i\in S} x_i \;\; \forall S \in N_E\}$$

Proof : This follows easily from lemma 3.2.7 and theorem 3.2.8 and the fact that an mcse is transformed into an mcst by the transition from mcse problem to associated mcst problem. □

Because the associated mcst problem does not depend on an mcse and the irreducible core of an mcst problem does not depend on an mcst, the following corollary holds.

Corollary 3.2.10 The irreducible core of an mcse problem is independent of the mcse used to define it.

Accordingly, the irreducible core of an mcse problem $\mathcal{M}$ will be denoted by $\mathrm{IC}(\mathcal{M})$.

Next follows the proof that the irreducible core of an mcse problem coincides with the set of allocations generated by algorithm 3.1.3.

Lemma 3.2.11 Let $\mathcal{M}$ be an mcse problem. Then $D^{GK}(\mathcal{M}) \subseteq \mathrm{IC}(\mathcal{M})$.

Proof : Section 3.1 stated that for an mcse problem $\mathcal{M}$, the set $D^{GK}(\mathcal{M})$ is a subset of the core of the associated game $(N, c^{\mathcal{M}})$. The proof in the appendix 3.5 depends only on the weights of the edges in an mcse E'. Since E' is an mcse in all mcse problems $\mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E')$, it follows that $D^{GK}(\mathcal{M}) \subseteq \mathrm{Core}(c^{\mathcal{M}'})$ for all mcse problems $\mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E')$. Hence, $D^{GK}(\mathcal{M}) \subseteq \bigcap_{\mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E')} \mathrm{Core}(c^{\mathcal{M}'}) = \mathrm{IC}(\mathcal{M})$. □

In order to prove the reverse inclusion, use the following lemma.

Lemma 3.2.12 Let $\mathcal{T} = \langle N, *, w \rangle$ be an mcst problem and let $\langle N^*, T \rangle$ be an mcst for $\mathcal{T}$. Then Bird's tree allocation β^T lies in the set $D^{GK}(\mathcal{T})$.

Proof : The number of edges in T equals $n := |N|$. Consider any sequence $\mathcal{E} = (e^1, \ldots, e^n)$ of edges obtained by ordering the edges of T by non-decreasing cost. Define $\mathcal{F} = (f^1, \ldots, f^n)$ by

$$f_i^t := \begin{cases} 1 & \text{if } e^t \text{ is the first edge on the path in } \langle N^*, T \rangle \text{ from } i \text{ to the source,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{F} \in V^{\mathcal{E}}(\mathcal{T})$ and that $\beta^T = x^{\mathcal{E}, \mathcal{F}} \in D^{\mathcal{E}} = D^{GK}(\mathcal{T})$ □

Theorem 3.2.13 Let $\mathcal{T}$ be an mcst problem. Then $D^{GK}(\mathcal{T}) = \mathrm{IC}(\mathcal{T})$.

Proof : It follows from lemma 3.2.11 that we only have to prove $\mathrm{IC}(\mathcal{T}) \subseteq D^{GK}(\mathcal{T})$. Bird (1976) proved $\mathrm{IC}(\mathcal{T})$ is the convex hull of the set of all Bird tree allocations of the mcst problem $\widetilde{\mathcal{T}}$, with reduced weight function defined by equation (3.2.3). By proposition 3.2.12, these Bird tree allocations lie in $D^{GK}(\widetilde{\mathcal{T}})$.

Aarts and Driessen (1993) proved that the edges in an mcst of $\mathcal{T}$ have the same weight in $\widetilde{\mathcal{T}}$ as in $\mathcal{T}$and that an mcst of $\mathcal{T}$is an mcst of $\widetilde{\mathcal{T}}$. Because the set D^{GK} is obtained by considering only the weights of edges in an mcst, it follows that $D^{GK}(\tilde{\mathcal{T}}) = D^{GK}(\mathcal{T})$. Hence, all Bird tree allocations lie in $D^{GK}(\mathcal{T})$.

Moreover $D^{GK}(\mathcal{T})$ is convex, hence

$$\mathrm{IC}(\mathcal{T}) = \text{conv hull}\{\beta^T \mid T \text{ is an mcst of } \tilde{\mathcal{T}}\} \subseteq D^{GK}(\tilde{\mathcal{T}}) = D^{GK}(\mathcal{T}),$$

which concludes the proof. □

Corollary 3.2.14 Let $\mathcal{M} = \langle N, *, w, E\rangle$ be an mcse problem. Then

$$D^{GK}(\mathcal{M}) = \mathrm{IC}(\mathcal{M}).$$

Proof : By lemma 3.2.11, we have only to prove that $D^{GK}(\mathcal{M}) \supseteq \mathrm{IC}(\mathcal{M})$. Suppose $x \in \mathrm{IC}(\mathcal{M})$ and let $\mathcal{T} = \langle N_E, *_E, w_E\rangle$ be the mcst problem associated with $\mathcal{M}$. Proposition 3.2.9 states that the vector $y \in \mathbf{R}^{N_E}$, defined by $y_S = \sum_{i \in S} x_i$ for all $S \in N_E$, lies in $\mathrm{IC}(\mathcal{T})$, which by theorem 3.2.13 equals $D^{GK}(\mathcal{T})$. Hence, there exists a sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ of edges leading to an mcst of $\mathcal{T}$ and a sequence $\mathcal{F} = (f^1, \ldots, f^\tau)$ of fraction vectors valid for $\mathcal{E}$, such that $y = x^{\mathcal{E},\mathcal{F}}$. Now for each edge $e^t = \{C_i, C_j\}$, there exists an edge $\tilde{e}^t$ with same weight in the weighted graph $\langle N^*, E_{N^*}, w\rangle$, which connects the components C_i and C_j. Hence, $\tilde{\mathcal{E}} = (\tilde{e}^1, \ldots, \tilde{e}^\tau)$ is a sequence leading to an mcse of $\mathcal{M}$. Define $\tilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^\tau)$ by

$$\tilde{f}_i^t = \begin{cases} f_{C_i}^t \frac{x_i}{y_{C_i}} & \text{if } y_{C_i} > 0 \\ 0 & \text{if } y_{C_i} = 0. \end{cases}$$

for all t and all $i \in N$, where C_i is the component containing i. Then $\tilde{\mathcal{F}}$ is valid for $\tilde{\mathcal{E}}$. Moreover, for any player i, $x_i^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}} = 0$ if $y_{C_i} = 0$, but then also $x_i = 0$, and if $y_{C_i} \neq 0$, then

$$x_i^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}} = \sum_{t=1}^{\tau} f_{C_i}^t \frac{x_i}{y_{C_i}} = \frac{x_i}{y_{C_i}} \sum_{t=1}^{\tau} f_{C_i}^t = x_i \frac{y_{C_i}}{y_{C_i}} = x_i.$$

Hence, $x = x^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}} \in D^{GK}(\mathcal{M})$, which completes the proof. □

A method has been provided (algorithm 3.1.3 together with equation 3.1.2) to compute the irreducible core of mcse problems. In order to obtain the whole core of the corresponding games, one needs usually to use weights of edges that are not used in any minimum-cost spanning extension. In general, it is still an open problem to compute the whole core of an mcse game directly from the weights of the edges, even if the attention is restricted to mcst games. Aarts (1992) and (1994) computed the core of mcst games for the case in which there is an mcst which is a chain—i.e. a tree with only two leaves.

3.3 The equal-remaining-obligations rule

In most cases, the irreducible core of an mcse problem $\mathcal{M}$ contains a continuum of allocations. If the objective is to choose a division of the cost, a one-point solution might be a better option.

The *equal-remaining-obligations rule* (henceforth *ERO rule*), suggested by Jos Potters, is a one-point refinement of the irreducible core and can be constructed according to the following idea.

In an mcse problem $\mathcal{M} = \langle N, *, w, E\rangle$, if player i is connected to the source, i has no need to construct anything and hence is unwilling to participate in the costs of edges. On the other hand, if he is not in the component of the source, i needs to be connected to the source. Because every component that is not connected to the source needs at least one edge to get connected to the source and constructing one edge per component not yet connected to the source is sufficient to connect every player to the source, we assume a component C that is not connected to the source has to pay one edge, or more precisely, fractions of edges summing up to 1. Moreover, the initial obligation of a component is shared equally by the players in this component. This leads to the *initial obligation* o_i of a player i, defined by

$$o_i := \begin{cases} \frac{1}{|C_i|} & \text{if } * \notin C_i \\ 0 & \text{if } * \in C_i \end{cases} \tag{3.3.1}$$

where C_i is the component of $\langle N^*, E\rangle$ containing player i.

For a sequence $\mathcal{F} = (f^1, \ldots, f^\tau)$ of fraction vectors, after a stage $t \leq \tau$ a player $i \in N$ has paid fractions of edges to a total of $\sum_{s \leq t} f_i^s$, while i's initial obligation was o_i. Hence player i's *remaining obligation* o_i^t satisfies

$$o_i^t = o_i - \sum_{s \leq t} f_i^s = o_i^{t-1} - f_i^t. \tag{3.3.2}$$

Now allocate the next edge e^t in such a way that the remaining obligations of the players are constant over components of the graph constructed by adding e^t.

Formally, one gets the following extension of algorithm 3.1.3.

Algorithm 3.3.1 (Equal-remaining-obligations rule)

input : an mcse problem $\mathcal{M}$

output : an mcse and the allocation ERO($\mathcal{M}$)

1. Given $\mathcal{M} \equiv \langle N, *, w, E\rangle$, define

$$\begin{array}{rcll} t &=& 0 & \text{the initial stage,} \\ \tau &=& |N^*/E| - 1 & \text{the number of stages,} \\ E^0 &=& \emptyset & \text{the initial edge set.} \end{array}$$

2. While $t < \tau$, do steps 3 to 6

3. $t := t+1$.

4. At stage t, given E^{t-1}, choose an edge $e^t \notin E \cup E^{t-1}$ such that the graph $\langle N^*, E \cup E^{t-1} \cup \{e^t\}\rangle$ contains no more cycles than the graph $\langle N^*, E \cup E^{t-1}\rangle$, and which is the cheapest edge with this property.

5. If $C^t = C_i^{t-1} \cup C_j^{t-1}$ is the connected component just formed by connecting the components C_i^{t-1} and C_j^{t-1} of the graph $\langle N^*, E \cup E^{t-1}\rangle$ with the edge $e^t = \{i, j\}$, define the fraction vector $f^t = (f_k^t)_{k \in N}$ by

$$f_k^t = \begin{cases} \frac{1}{|C_i^{t-1}|} - \frac{1}{|C^t|} & \text{if } k \in C_i^{t-1} \text{ and } * \notin C^t, \\ \frac{1}{|C_j^{t-1}|} - \frac{1}{|C^t|} & \text{if } k \in C_j^{t-1} \text{ and } * \notin C^t, \\ \frac{1}{|C_i^{t-1}|} & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1}, \\ \frac{1}{|C_j^{t-1}|} & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1}, \\ 0 & \text{otherwise.} \end{cases}$$

6. Define $E^t := E^{t-1} \cup \{e^t\}$.

7. E^τ is the mcse we sought. As before, denote $\mathcal{E} = (e^1, \ldots, e^\tau)$ the sequence of edges constructed.

8. Define $\mathrm{ERO}^{\mathcal{E}}(\mathcal{M}) := \sum_{t=1}^{\tau} f^t w(e^t)$.

Example 3.3.2 Applied to the mcse problem of example 3.1.2, this algorithm generates successively

- edge $\{1,2\}$, of which players 1 and 2 each pay $1 - \frac{1}{2} = \frac{1}{2}$,
- edge $\{2,3\}$, of which player 3 pays $1 - \frac{1}{3} = \frac{2}{3}$ and players 1 and 2 each pay $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$,
- edge $\{3,4\}$, of which players 1, 2 and 3 each pay $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, while players 4 and 5 pay $\frac{1}{2} - \frac{1}{5} = \frac{3}{10}$
- and, finally, edge $\{2,*\}$, to which each player contributes $\frac{1}{5}$.

This yields the allocation $\frac{1}{3}(101, 101, 116, 96, 96)$.

Generically, the choice of edge in step 4 is unique, but even in the case that the sequence $\mathcal{E}$ is not uniquely defined, this algorithm yields only one allocation, independent of the choice of edges made. This contrasts with Bird's tree-allocation rule, which may associate a different allocation with each mcst of an mcst problem.

Proposition 3.3.3 For any two sequences of edges $\mathcal{E}$ and $\tilde{\mathcal{E}}$ chosen by the algorithm 3.3.1 applied to an mcse problem $\mathcal{M}$,

$$\mathrm{ERO}^{\mathcal{E}}(\mathcal{M}) = \mathrm{ERO}^{\tilde{\mathcal{E}}}(\mathcal{M}).$$

Since the proof is similar to the proof of proposition 3.1.10, it will not be given here. This proposition allows us to define

$$\mathrm{ERO}(\mathcal{M}) := \mathrm{ERO}^{\mathcal{E}}(\mathcal{M})$$

for any sequence $\mathcal{E}$ constructed by algorithm 3.3.1. Clearly, the fraction vectors constructed are valid for the edges constructed; the ERO rule is thus a refinement of the irreducible core.

The ERO rule got its name from the following theorem.

Theorem 3.3.4 The algorithm 3.3.1 has the property that after each stage t, in each component C of the graph $\langle N^*, E \cup E^t \rangle$, every player k in the component C has the same remaining obligation

$$o_k^t = \begin{cases} \frac{1}{|C|} & \text{if } * \notin C, \\ 0 & \text{if } * \in C. \end{cases} \tag{3.3.3}$$

Proof : The proof goes by induction on the stage t.

1. After stage zero, for k in a component C of $\langle N^*, E \rangle$,

$$o_k^t = o_k - 0 = \begin{cases} \frac{1}{|C|} & \text{if } * \notin C, \\ 0 & \text{if } * \in C. \end{cases}$$

2. Suppose equation (3.3.3) holds after stage $t-1$. Let $C^t = C_i^{t-1} \cup C_j^{t-1}$ be the connected component formed at stage t by connecting the components C_i^{t-1} and C_j^{t-1} of the graph $\langle N^*, E \cup E^{t-1} \rangle$ with the edge $e^t = \{i, j\}$. Let C_k^t and C_k^{t-1} be the components of player k in $\langle N^*, E \cup E^t \rangle$ and $\langle N^*, E \cup E^{t-1} \rangle$. Then

$$\begin{aligned} o_k^t &= o_k^{t-1} - f_k^t \\ &= \begin{cases} \frac{1}{|C_k^{t-1}|} - f_k^t & \text{if } k \notin C_*^{t-1} \\ 0 - 0 & \text{if } k \in C_*^{t-1} \end{cases} \\ &= \begin{cases} \frac{1}{|C_i^{t-1}|} - (\frac{1}{|C_i^{t-1}|} - \frac{1}{|C^t|}) & \text{if } k \in C_i^{t-1} \text{ and } * \notin C^t \\ \frac{1}{|C_j^{t-1}|} - (\frac{1}{|C_j^{t-1}|} - \frac{1}{|C^t|}) & \text{if } k \in C_j^{t-1} \text{ and } * \notin C^t \\ \frac{1}{|C_i^{t-1}|} - \frac{1}{|C_i^{t-1}|} & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1} \\ \frac{1}{|C_j^{t-1}|} - \frac{1}{|C_j^{t-1}|} & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1} \\ \frac{1}{|C_k^{t-1}|} & \text{if } k \notin C^t \cup C_*^{t-1} \\ 0 & \text{if } k \in C_*^{t-1} \end{cases} \end{aligned}$$

$$
= \begin{cases} \frac{1}{|C^t|} & \text{if } k \in C^t \not\ni * \\ 0 & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1} \\ 0 & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1} \\ \frac{1}{|C_k^{t-1}|} & \text{if } k \notin C^t \cup C_*^{t-1} \\ 0 & \text{if } * \in C_k^{t-1} \end{cases}
$$

$$
= \begin{cases} \frac{1}{|C_k^t|} & \text{if } * \notin C_k^t \\ 0 & \text{if } * \in C_k^t. \end{cases}
$$

Hence equation (3.3.3) holds after stage t as well. This completes the proof. □

3.4 Axiomatic characterizations

Sections 3.2 and 3.3 introduced the irreducible core and the equal-remaining-obligations rule for mcse problems. We will axiomatically characterize these rules in this section.

In contrast with the characterizations in chapters 2 and 4, a solution here consists only of a set of allocations. This is the case because both the irreducible core and the ERO rule are independent of the set of edges constructed.

An *allocation* of an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ is a vector $x \in \mathbf{R}_+^N$ that satisfies $\sum_{i \in N} x_i \geq c^{\mathcal{M}}(N)$. In effect, an allocation is a vector that allocates at least the cost of a minimum-cost spanning extension to the players.

Properties that an allocation x of an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ can satisfy are

Definition 3.4.1

Eff x is *efficient* if

$$\sum_{i \in N} x_i = c^{\mathcal{M}}(N).$$

MC x has the *minimal contribution property* if every component that does not contain the source contributes at least the cost of a minimum-cost edge that connects two components. In formula : for each component $C \in N^*/E$ that does not contain the source,

$$\sum_{i \in C} x_i \geq \min\{w(e) \mid e \text{ connects two components of } \langle N^*, E\rangle\}.$$

FSC x has the *free-for-source-component* property if $x_i = 0$ for all i in the component of the source in the graph $\langle N^*, E\rangle$.

The minimal contribution property and the free-for-source-component property are motivated as follows: every component that has to be connected to the source has to contribute at least the cost of an edge, and the component of the source should not contribute, because it is already connected.

A *solution* of mcse problems is a map ψ assigning to every mcse problem a set of allocations.

Definition 3.4.2

NE A solution ψ is said to be *non-empty* if

$$\psi(\mathcal{M}) \neq \emptyset \text{ for all mcse problems } \mathcal{M}.$$

A solution ψ can be said to be *efficient* and satisfies the *minimal contribution property* or the *free-for-source-component property* if for all $\mathcal{M}$ all elements of the solution $\psi(\mathcal{M})$ satisfy the corresponding property.

Definition 3.4.3 Given an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ and an edge $e \notin E$ that connects two components of $\langle N^*, E\rangle$, define the *edge-reduced mcse problem*

$$\mathcal{M}^e = \langle N, *, w, E \cup \{e\}\rangle.$$

The next three properties use edge-reduced mcse problems, with as extra edge an edge which constructs a new component if it is adjoined to the graph $\langle N^*, E\rangle$, and which has minimum cost among all edges with this property.

Definition 3.4.4

Loc ψ is *local* if for all $\mathcal{M}$, for all $x \in \psi(\mathcal{M})$, for each minimum-cost edge e that when added to $\langle N^*, E\rangle$ constructs a new component C, there exists an $\tilde{x} \in \mathbf{R}^C$ such that

$$(\tilde{x}, x^{N\setminus C}) \in \psi(\mathcal{M}^e).$$

In effect, this axiom requires that when a minimum-cost edge is added, this does not influence the allocation to players that are not in the component constructed by adding this edge.

ECons ψ is *minimum-cost-edge consistent* if for all $\mathcal{M} \equiv \langle N, *, w, E\rangle$, for all allocations $x \in \psi(\mathcal{M})$, for each minimum-cost edge e that when added to $\langle N^*, E\rangle$ constructs a new component C, for each $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, it holds that

$$x - x^{e,\alpha} \in \psi(\mathcal{M}^e),$$

where $x^{e,\alpha} := (\alpha w(e), 0^{N\setminus C})$.

This axiom means that when a minimum-cost edge is added, the savings obtained by not having to construct this edge can be allocated arbitrarily over the players that are in the component constructed by adding this edge. Obviously, edge consistency implies locality.

CECons ψ is *converse minimum-cost-edge consistent* if for all $\mathcal{M} \equiv \langle N, *, w, E\rangle$, for every minimum-cost edge e that when added to $\langle N^*, E\rangle$ constructs a new component C, for every $x \in \mathbf{R}^N$ that satisfies

a MC,

b FSC,

c $\alpha w(e) \leq x^C$ implies $x - x^{e,\alpha} \in \psi(\mathcal{M}^e)$ for all $\alpha \in \Delta^C$,

it holds that

$$x \in \psi(\mathcal{M}).$$

This axiom requires that if adding an allocation to the solution does not destroy the MC, FSC and ECons properties, then it should be part of the solution. In effect, it requires the solution to be the largest solution that satisfies the other properties.

Proposition 3.4.5 The irreducible core satisfies the properties NE, MC, Eff, FSC, ECons and CECons.

Proof : Because of the coincidence of the irreducible core with the set D^{GK}, the irreducible core is non-empty for any mcse problem : there are always valid sequences of fraction vectors for any sequence of edges constructed by the algorithm 3.1.3. It satisfies the minimum contribution property because every component that does not contain the source has to pay for fractions of edges that total one, so it contributes at least the minimum cost of an edge that connects two components. That it is efficient has been proved in lemma 3.1.4. By construction, it is clear that D^{GK} satisfies FSC.

To prove edge consistency, take an mcse problem $\mathcal{M}$ and suppose $x \in \mathrm{IC}(\mathcal{M}) = D^{GK}(\mathcal{M})$. For each minimum-cost edge e that when added to $\langle N^*, E\rangle$ constructs a new component C, there exists a sequence $\mathcal{E} = (e^1, e^2, \ldots, e^\tau)$ starting with the edge $e^1 = e$, that is constructed by the algorithm 3.1.3. Because the set $D^{GK}(\mathcal{M})$ is independent of the sequence of edges constructed, there exists a sequence $\mathcal{F} = (f^1, \ldots, f^\tau) \in V^{\mathcal{E}}$ such that $x = x^{\mathcal{E},\mathcal{F}}$. For any $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, define $\alpha^1 \in \Delta^C$ by $\alpha^1_i := f^1_i$ for all $i \in C$. Then

$$\begin{cases} \displaystyle\sum_{i\in C}(\alpha_i - \alpha^1_i) &= 0, \\ (\alpha - \alpha^1)w(e^1) &\leq \displaystyle\sum_{t=2}^{\tau} f^t w(e^t), \\ f^t w(e^t) &\geq 0 \qquad \text{for } t \geq 2. \end{cases}$$

Hence there exist vectors $\alpha^2, \ldots, \alpha^\tau \in \mathbf{R}^C$ satisfying

$$\left\{\begin{array}{rcll} \alpha_i^t & \leq & f_i^t & \text{for } t \geq 2 \text{ and for all } i \in C, \\ \sum_{i \in C} \alpha_i^t & = & 0 & \text{for } t \geq 2, \\ (\alpha - \alpha^1)w(e^1) & = & \sum_{t=2}^{\tau} \alpha^t w(e^t). & \end{array}\right. \tag{3.4.1}$$

E.g. take α^t the projection of f^t on the hyperplane with coordinates zero, along the line through the points $\sum_{s=2}^{\tau} f^s w(e^s)$ and $(\alpha - \alpha^1)w(e^1)$, i.e.

$$\alpha^t = f^t - \frac{\sum_{i \in C} f_i^t}{\sum_{i \in C} \sum_{s=2}^{\tau} f_i^s w(e^s)} \left(\sum_{s=2}^{\tau} f^s w(e^s) - (\alpha - \alpha^1)w(e^1)\right)$$

for all $t \geq 2$. Rewriting the last equation of system (3.4.1), we obtain

$$\alpha w(e) = \sum_{t=1}^{\tau} \alpha^t w(e^t).$$

It follows that $x - x^{e,\alpha} = \sum_{t=2}^{\tau} (f^t - \alpha^t) w(e^t)$. The first two equations of system (3.4.1) imply that the sequence $(f^2 - \alpha^2, \ldots, f^\tau - \alpha^\tau)$ is valid for $(e^2, \ldots, e^\tau)$. Hence, $x - x^{e,\alpha} \in \mathcal{M}^e$.

Because edge consistency implies locality, the irreducible core satisfies locality as well.

To prove that the irreducible core satisfies CECons, take an mcse problem $\mathcal{M}$, take a minimum-cost edge e that when added to $\langle N^*, E \rangle$ constructs a new component C, take an $x \in \mathbf{R}^N$ that satisfies

a MC,

b FSC,

c $\alpha w(e) \leq x^C$ implies $x - x^{e,\alpha} \in \mathrm{IC}(\mathcal{M}^e)$ for all $\alpha \in \Delta^C$.

We have to prove that $x \in \mathrm{IC}(\mathcal{M})$.

Denote by C_1 and C_2 the two components that are joined by e. The allocation x satisfies FSC; hence if one of these components (say C_1) contains the source, $x_i = 0$ for $i \in C_1$ and an $\alpha \in \Delta^C$ with $\alpha \leq x^C$ satisfies $\alpha_i = 0$ for $i \in C_1$. For such an α (which exists), there exists a sequence $(e^2, \ldots, e^\tau)$ constructed by the algorithm 3.1.3 applied to the problem $\mathcal{M}^e$ and a sequence $(f^2, \ldots, f^\tau) \in V^{(e^2, \ldots, e^\tau)}$ such that $x - x^{e,\alpha} = x^{(e^2, \ldots, e^\tau), (f^2, \ldots, f^\tau)}$. So with f defined by

$$f_k := \left\{\begin{array}{ll} \alpha_k & \text{if } k \in C \\ 0 & \text{otherwise} \end{array}\right.$$

for all $k \in N$, it holds that $(f, f^2, \ldots, f^\tau)$ is valid for the sequence $(e, e^2, \ldots, e^\tau)$ and

$$x = x^{(e, e^2, \ldots, e^\tau), (f, f^2, \ldots, f^\tau)} \in \psi(\mathcal{M}).$$

If neither of the two components contain the source, then by the minimal contribution property, both components contribute at least $w(e)$ in the allocation x. On the other hand, both together contribute $x(C)$, so there exists an $a^1 \in [0,1]$, such that

$$\begin{cases} x(C_1) &= a^1 w(e) + (1-a^1)(x(C) - w(e)) \\ x(C_2) &= (1-a^1)w(e) + a^1(x(C) - w(e)) \end{cases}$$

Define $\alpha \in \Delta^C$, by

$$\alpha_i = \begin{cases} a^1 x_i / x(C_1) & \text{if } i \in C_1, \\ (1-a^1)x_i/x(C_2) & \text{if } i \in C_2. \end{cases}$$

Then, $\alpha w(e) \leq x^C$. Hence, there exists an mcse $\{e^2, \ldots, e^\tau\}$ and a sequence $(f^2, \ldots, f^\tau) \in V^{(e^2,\ldots,e^\tau)}(\mathcal{M})$ such that $x - x^{e,\alpha} = \sum_{t=2}^{\tau} f^t w(e^t)$. Define $a_1^2, \ldots, a_1^\tau$ by $a_1^t = (1-a^1)\sum_{i\in C} f_i^t$ and $a_2^2, \ldots, a_2^\tau$ by $a_2^t = a^1 \sum_{i \in C} f_i^t$. Then

$$\begin{cases} a_1^t + a_2^t &= \sum_{i\in C} f_i^t & \text{for all } t \geq 2 \\ a^1 + \sum_{t=2}^{\tau} a_1^t &= a^1 + (1-a^1)\sum_{t=2}^{\tau}\sum_{i\in C} f_i^t & = 1 \\ 1 - a^1 + \sum_{t=2}^{\tau} a_2^t &= 1 - a^1 + a^1 \sum_{t=2}^{\tau}\sum_{i\in C} f_i^t & = 1 \end{cases}$$

Defining $g^1 = (\alpha, 0^{N\setminus C})$ and

$$g_i^t = \begin{cases} a_1^t x_i / x(C_1) & \text{if } i \in C_1 \\ a_2^t x_i / x(C_2) & \text{if } i \in C_2 \end{cases}$$

we see that $(g^1, \ldots, g^\tau) \in V^{(e,e^2,\ldots,e^\tau)}(\mathcal{M})$. Furthermore,

$$\begin{aligned} x(C_1) - a^1 w(e) &= (1-a^1)(x(C) - w(e)) \\ &= (1-a^1)\sum_{t=2}^{\tau}\sum_{i\in C} f_i^t w(e^t) \\ &= \sum_{t=2}^{\tau} a_1^t w(e^t). \end{aligned}$$

This implies that for $i \in C_1$,

$$\begin{aligned} x_i &= \frac{x_i}{x(C_1)} x(C_1) \\ &= \frac{x_i}{x(C_1)}(a^1 w(e) + \sum_{t=2}^{\tau} a_1^t w(e^t)) \\ &= g_i^1 w(e) + \sum_{t=2}^{\tau} g_i^t w(e^t). \end{aligned}$$

Similarly for $i \in C_2$. Hence, $x = x^{(e,e^2,\ldots,e^\tau),(g^1,\ldots,g^\tau)}$ is in the irreducible core of $\mathcal{M}$. This implies that the irreducible core satisfies converse edge consistency. □

Lemma 3.4.6 If a solution of mcse problems ϕ satisfies MC, FSC and ECons, and a solution of mcse problems ψ satisfies NE, FSC and CECons, then $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ for each mcse problem $\mathcal{M}$.

Proof : We prove the lemma for any mcse problem $\mathcal{M}$ by induction on the number of components of the graph $\langle N^*, E\rangle$. First, consider first an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$, where the graph $\langle N^*, E\rangle$ is connected. Then by FSC, $x = 0$ for any $x \in \phi(\mathcal{M})$ and for any $x \in \psi(\mathcal{M})$. By NE, there has to be an $x \in \psi(\mathcal{M})$, hence

$$\phi(\mathcal{M}) \subseteq \{0\} = \psi(\mathcal{M}).$$

Now suppose the lemma holds for every mcse problem $\mathcal{M}$ with $p-1$ components in the graph $\langle N^*, E\rangle$. Take an mcse problem $\mathcal{M}$ such that $\langle N^*, E\rangle$ has p components and take $x \in \phi(\mathcal{M})$. By ECons of ϕ, for any minimum-cost edge that constructs a new component C when added to $\langle N^*, E\rangle$, for each $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, it holds that

$$x - x^{e,\alpha} \in \phi(\mathcal{M}^e) \subseteq \psi(\mathcal{M}^e),$$

where the inclusion holds by the induction hypothesis. Because x satisfies MC, it holds that $x \in \psi(\mathcal{M})$ by CECons of ψ. Hence $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$. □

Theorem 3.4.7 The unique solution of mcse problems that satisfies NE, MC, FSC, ECons and CECons is the irreducible core.

Proof : By proposition 3.4.5, the irreducible core has these properties. By lemma 3.4.6, if two solutions satisfy these properties, they contain each other and hence they have to coincide. □

To characterize the ERO rule, we introduce two other properties.

Definition 3.4.8

ET a solution ψ satisfies *equal treatment* if for every mcse problem $\mathcal{M}$, for all $x \in \psi(\mathcal{M})$, for each component C of the original graph $\langle N^*, E\rangle$, and for all players $i, j \in C$,

$$x_i = x_j.$$

IPCons A solution ψ is *inversely proportional consistent* if for every mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$, for every minimum-cost edge e that when added to $\langle N^*, E\rangle$ connects two components C_1 and C_2, neither of which contains the source, for all $x \in \psi(\mathcal{M})$, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that

$$|C_1| \sum_{i \in C_1} (x_i - \tilde{x}_i) = |C_2| \sum_{i \in C_2} (x_i - \tilde{x}_i).$$

Equal treatment is motivated by the idea that players in the same component need the same connections, hence should be allocated the same amount. Inversely proportional consistency means that if two components are connected by a minimum-cost edge, they contribute amounts to the cost of this edge which are inversely proportional to their number of elements. This is motivated by the idea that a bigger component has apparently already constructed more edges in E, so should pay less for the edge under consideration.

Theorem 3.4.9 The unique solution of mcse problems that satisfies NE, FSC, Loc, Eff, ET and IPCons is the ERO rule. Here, the ERO rule is identified with the solution that assigns the singleton $\{\text{ERO}(\mathcal{M})\}$ to mcse problem $\mathcal{M}$.

Proof : First we prove that the ERO rule satisfies the required properties. That the ERO rule satisfies the properties NE, MC, FSC, Loc and Eff is a consequence of its being a refinement of the irreducible core. That is satisfies equal treatment is also easy to see. To prove it satisfies IPCons, take an mcse problem $\mathcal{M}$, and a minimum-cost edge e^1 connecting the components C_1 and C_2, neither of which contains the source, into a component C. Then there exists a sequence of edges $\mathcal{E} = (e^1, \ldots, e^\tau)$ starting with e^1 and a sequence of fraction vectors $\mathcal{F} = (f^1, \ldots, f^\tau)$ constructed by the algorithm 3.3.1 such that $\text{ERO}(\mathcal{M}) = x^{\mathcal{E},\mathcal{F}}$. Moreover, by definition of the algorithm, $x^{(e^2,\ldots,e^\tau),(f^2,\ldots,f^\tau)} = \text{ERO}(\mathcal{M}^e)$. Because e connects two components that do not contain the source,

$$x_k^{\mathcal{E},\mathcal{F}} - x_k^{(e^2,\ldots,e^\tau),(f^2,\ldots,f^\tau)} = \begin{cases} w(e)(\frac{1}{|C_1|} - \frac{1}{|C|}) & \text{if } k \in C_1, \\ w(e)(\frac{1}{|C_2|} - \frac{1}{|C|}) & \text{if } k \in C_2, \\ 0 & \text{if } k \notin C. \end{cases}$$

Then

$$\sum_{k \in C_1} (\text{ERO}(\mathcal{M}) - \text{ERO}(\mathcal{M}^e)) = 1 - \frac{|C_1|}{|C|} = \frac{|C_2|}{|C|}$$

and similarly,

$$\sum_{k \in C_2} (\text{ERO}(\mathcal{M}) - \text{ERO}(\mathcal{M}^e)) = \frac{|C_1|}{|C|}.$$

Hence

$$|C_1| \sum_{k \in C_1} (\text{ERO}_k(\mathcal{M}) - \text{ERO}_k(\mathcal{M}^e)) = \frac{|C_1||C_2|}{|C|} = |C_2| \sum_{k \in C_2} (\text{ERO}_k(\mathcal{M}) - \text{ERO}_k(\mathcal{M}^e)).$$

To prove uniqueness, suppose a solution ψ satisfies these six properties. We prove $\psi(\mathcal{M}) = \{\text{ERO}(\mathcal{M})\}$ by induction on the number of components of the graph $\langle N^*, E \rangle$.

Let $\langle N^*, E \rangle$ have one component. By FSC, $x_i(\mathcal{M}) = 0 = \text{ERO}_i(\mathcal{M})$ for all $i \in N$ and all $x \in \psi(\mathcal{M})$.

Suppose $\psi(\mathcal{M}) = \{\text{ERO}(\mathcal{M})\}$ for all mcse problems $\mathcal{M}$ such that $\langle N^*, E \rangle$ has less than p components. Consider an mcse problem $\mathcal{M}$ such that $\langle N^*, E \rangle$ has p components.

Take a minimum-cost spanning edge e that connects two components C_1 and C_2 into a new component C in $\langle N^*, E\rangle$. By ET of ψ applied to $\mathcal{M}$ and $\mathcal{M}^e$, for all $x \in \psi(\mathcal{M})$, for all $\tilde{x} \in \psi(\mathcal{M}^e)$, for all $i, j \in C_1$ we have

$$x_j - \tilde{x}_j = x_i - \tilde{x}_i =: \delta_1(x, \tilde{x})$$

and for all $i, j \in C_2$ we have

$$x_j - \tilde{x}_j = x_i - \tilde{x}_i =: \delta_2(x, \tilde{x}).$$

Moreover, by FSC of ψ, if C_1 contains the source, then $\delta_1(x, \tilde{x}) = 0$ and by locality and efficiency, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that $\delta_2(x, \tilde{x}) = \frac{w(e)}{|C_2|}$. If C_2 contains the source, then similarly one proves that $\delta_1(x, \tilde{x})$ and $\delta_2(x, \tilde{x})$ are uniquely determined. If neither of C_1 and C_2 contain the source, then by IPCons, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that

$$|C_1| \sum_{k \in C_1} \delta_1(x, \tilde{x}) = |C_2| \sum_{k \in C_2} \delta_2(x, \tilde{x}),$$

and by locality and efficiency,

$$\sum_{i \in C_1} \delta_1(x, \tilde{x}) + \sum_{i \in C_2} \delta_2(x, \tilde{x}) = \sum_{i \in C} (x_i - \tilde{x}_i) = w(e).$$

Hence,

$$\delta_1(x, \tilde{x}) = \frac{w(e)}{|C_1|^2} \qquad \text{and} \qquad \delta_2(x, \tilde{x}) = \frac{w(e)}{|C_2|^2}.$$

So whether C_1 or C_2 contain the source or not, the numbers $\delta_1(x, \tilde{x})$ and $\delta_2(x, \tilde{x})$ are uniquely determined and independent of x and $\tilde{x}$. Now the ERO rule also satisfies the six properties and so has these same numbers $\delta_1(x, \tilde{x})$, $\delta_2(x, \tilde{x})$ characterizing the difference between ERO($\mathcal{M}$) and ERO($\mathcal{M}^e$). The induction hypothesis then implies

$$x - \delta_1(x, \tilde{x}) 1_{C_1} - \delta_2(x, \tilde{x}) 1_{C_2} = \tilde{x} = \mathrm{ERO}(\mathcal{M}^e) = \mathrm{ERO}(\mathcal{M}) - \delta_1(x, \tilde{x}) 1_{C_1} - \delta_2(x, \tilde{x}) 1_{C_2}$$

and so $x = \mathrm{ERO}(\mathcal{M})$. □

It is still an open problem whether the properties used to characterize the irreducible core in theorem 3.4.7 and the equal-remaining-obligations value in theorem 3.4.9 are logically independent, but we conjecture that they are.

3.5 Appendix

In this appendix, we prove theorem 3.1.8 and proposition 3.1.10. First, we need a few lemmas.

Lemma 3.5.1 For all sequences $\mathcal{E}$ chosen by algorithm 3.1.3 the constructed extension E^τ is an mcse for the problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$.

Proof : There are $\tau + 1 = |N^*/E|$ components in $\langle N^*, E\rangle$, at each stage two components are connected, so after stage τ, the resulting graph is connected and no new cycles have been introduced. Assume that the extension $\langle N^*, E \cup E^\tau\rangle$ constructed is not minimal in cost, i.e. there exists a set of edges $\tilde{E}$ such that $|\tilde{E}| = \tau$ and $\langle N^*, E \cup \tilde{E}\rangle$ is a connected graph, and

$$\sum_{e \in \tilde{E}} w(e) < \sum_{e \in E^\tau} w(e). \tag{3.5.1}$$

Let the sequence $\tilde{\mathcal{E}} = (\tilde{e}^1, \ldots, \tilde{e}^\tau)$ consist of the edges in $\tilde{E}$ ordered by non-decreasing weight. Equation (3.5.1) implies there exists a smallest $t \le \tau$ such that $e^{t'} = \tilde{e}^{t'}$ for $1 \le t' < t$ and $e^t \neq \tilde{e}^t$. Because e^t is a minimum-cost edge that does not introduce a cycle in $\langle N^*, E \cup E^{t-1}\rangle = \langle N^*, E \cup \{\tilde{e}^1, \ldots, \tilde{e}^{t-1}\}\rangle$ and $\tilde{e}^t$ does not introduce a cycle in $\langle N^*, E \cup E^{t-1}\rangle$, it follows that $w(e^t) \le w(\tilde{e}^t)$. Consider the end points i and j of e^t. They have to be connected in $\langle N^*, E \cup \tilde{E}\rangle$, hence there exists a path from i to j in $\langle N^*, E \cup \tilde{E}\rangle$. But not all edges in this path can be present in the graph $\langle N^*, E \cup E^{t-1}\rangle$, otherwise e^t would introduce a cycle. Hence there is an edge $e \in \tilde{E}$ in this path that comes later in $\tilde{\mathcal{E}}$ than $\tilde{e}^t$, so e costs at least $w(\tilde{e}^t)$, which is at least $w(e^t)$. Now $E' := (\tilde{E} \setminus \{e\}) \cup \{e^t\}$ is a spanning extension of $\langle N^*, E\rangle$ such that E' does not cost more than $\tilde{E}$, and E' has one edge more in common with E^τ. Repeating this process enough times shows that E^τ does not cost more than $\tilde{E}$. This is a contradiction, hence the assumption that the algorithm 3.1.3 does not lead to an mcse is wrong. □

In order to prove that $D^{\mathcal{E}}$ is a subset of the core of the associated mcse game if $\mathcal{E}$ is constructed by algorithm 3.1.3, we need to compare the outcome of the algorithm 3.1.3 applied to related mcse problems.

Suppose we have an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ and an edge $e = \{i, j\}$ connecting the component C_* of the source with the component C_j of some player j in the graph $\langle N^*, E\rangle$. Define $\tilde{E} := E \cup \{e\}$. Consider the mcse problem $\widetilde{\mathcal{M}} = \langle N, *, w, \tilde{E}\rangle$. Distinguish the graphs, components, edges and allocations used in algorithm 3.1.3 applied to the problems $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ by giving those in the latter problem a tilde. With this setup, we prove two lemmata and a corollary, which we need to prove theorem 3.1.8.

Lemma 3.5.2 For every sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 3.1.3 applied to $\mathcal{M}$, one can find an $s \le \tau$ such that the sequence $\tilde{\mathcal{E}} \equiv (\tilde{e}^1, \ldots, \tilde{e}^{\tau-1}) := (e^1, \ldots, e^{s-1}, e^{s+1}, \ldots, e^\tau)$, obtained by deleting the edge e^s from $\mathcal{E}$, is a sequence of edges that can be obtained by algorithm 3.1.3 applied to $\widetilde{\mathcal{M}}$ and that satisfies

1. $(N^*/(E \cup E^t)) \setminus \{C_i^t, C_j^t\} = (N^*/(\tilde{E} \cup \tilde{E}^t)) \setminus \{\tilde{C}_i^t\}$ and $C_i^t \cup C_j^t = \tilde{C}_i^t$ for all $t < s$, that is, as long as $t < s$, the graphs $\langle N^*, E \cup E^t\rangle$ and $\langle N^*, \tilde{E} \cup \tilde{E}^t\rangle$ have the same components, except for the components of i and j in $\langle N^*, E \cup E^t\rangle$, which are connected to each other in $\langle N^*, \tilde{E} \cup \tilde{E}^t\rangle$.

2. $N^*/(E \cup E^t) = N^*/(\tilde{E} \cup \tilde{E}^{t-1}) \qquad$ for all $t \in \{s, \ldots, \tau\}$, that is, after stage s, the components of $\langle N^*, E \cup E^t\rangle$ coincide with the components

of $\langle N^*, \tilde{E} \cup \tilde{E}^{t-1}\rangle$ at the previous stage.

Proof : We prove the statements by induction on t. For $t = 0$: $\tilde{E} \cup \tilde{E}^0 = \tilde{E} = E \cup \{e\} = E \cup E^0 \cup \{e\}$, hence $\tilde{C}_i^t = C_i^t \cup C_j^t$ and $(N^*/(E \cup E^t)) \setminus \{C_i^t, C_j^t\} = (N^*/(\tilde{E} \cup \tilde{E}^t)) \setminus \{\tilde{C}_i^t\}$. Hence case 1 holds.

If case 1 holds at stage $t-1$, look at the effect of adding e^t to $\tilde{E}^{t-1}$. Two cases can occur.

a $e^t \neq e$ and adding the edge e^t does not introduce a cycle in the graph $\langle N^*, \tilde{E} \cup \tilde{E}^{t-1}\rangle$. Now e^t is a cheapest edge which does not introduce a cycle in $\langle N^*, E \cup E^{t-1}\rangle$ and any edge which does not introduce a cycle in $\langle N^*, \tilde{E} \cup \tilde{E}^{t-1}\rangle$, does not introduce a cycle in $\langle N^*, E \cup E^{t-1}\rangle$. Hence e^t is also a cheapest edge that does not introduce a cycle in $\langle N^*, \tilde{E} \cup \tilde{E}^{t-1}\rangle$. This means e^t is a legitimate choice for $\tilde{e}^t$. Consequently, case 1 still holds at stage t.

b $e^t = e$ or adding the edge e^t does introduce a cycle in the graph $\langle N^*, \tilde{E} \cup \tilde{E}^{t-1}\rangle$. Because e^t does not add any cycle in $\langle N^*, E \cup E^{t-1}\rangle$, this means e^t connects the components C_i^{t-1} and C_i^{t-1} of $\langle N^*, E \cup E^{t-1}\rangle$. Then $C_i^t = C_i^{t-1} \cup C_j^{t-1} = \tilde{C}_i^{t-1}$ and the other components are unchanged, so

$$\begin{aligned}(N^*/(E \cup E^t)) \setminus \{C_i^t\} &= (N^*/(E \cup E^{t-1})) \setminus \{C_i^{t-1}, C_j^{t-1}\} \\ &= (N^*/(\tilde{E} \cup \tilde{E}^{t-1})) \setminus \{\tilde{C}_i^{t-1}\}.\end{aligned}$$

Hence $N^*/(E \cup E^t) = N^*/(\tilde{E} \cup \tilde{E}^{t-1}))$, and case 2 holds for stage t.

Suppose case 2 holds for stage $t-1$. Then the edge e^t is a legitimate choice for $\tilde{e}^{t-1}$ (it does not introduce a cycle, and has minimal cost among the edges satisfying this). Suppose $e^t = \{k, l\}$. Hence, $C_k^t = C_k^{t-1} \cup C_l^{t-1} = \tilde{C}_k^{t-2} \cup \tilde{C}_l^{t-2} = \tilde{C}_k^{t-1}$, which implies $N^*/(E \cup E^t) = N^*/(\tilde{E} \cup \tilde{E}^{t-1})$, and case 2 holds for stage t as well.

So if the first stage at which case 2 holds is called s, we see that case 1 holds for $t < s$ and case 2 holds for $t \geq s$.

Note that $N^*/(E \cup E^\tau) = \{N^*\} = N^*/(\tilde{E} \cup \tilde{E}^{t-1})$, hence case 2 holds at stage τ, so $s \leq \tau$. □

Lemma 3.5.3 Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be as above, let $\mathcal{E}$ be a sequence of choices made by algorithm 3.1.3 applied to $\mathcal{M}$, and let $\mathcal{F}$ be valid for $\mathcal{E}$. Let the sequence $\widetilde{\mathcal{E}}$ and the stage s be as defined in lemma 3.5.2. Then there exist $(\tilde{t}_k)_{k \in N}$, such that the sequence $\widetilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^{\tau-1})$ defined by

$$\tilde{f}_k^t := \begin{cases} f_k^t & \text{if } t < \min\{\tilde{t}_k, s\} \\ \sum\limits_{t'=\tilde{t}_k}^{\tau} f_k^{t'} & \text{if } t = \tilde{t}_k < s \\ f_k^{t+1} & \text{if } \tilde{t}_k \geq t \geq s \\ 0 & \text{if } t > \tilde{t}_k \end{cases} \qquad \text{for all } t \in \{1, \ldots, \tau - 1\} \text{ and all } k \in N$$

is valid for $\tilde{\mathcal{E}}$. In formula : $\tilde{\mathcal{F}} \in V^{\tilde{\mathcal{E}}}(\widetilde{\mathcal{M}})$.

Proof : For all $k \in N$, define

$$\tilde{t}_k := \min\{t \mid \text{there exists a path from } k \text{ to } * \text{ in } \langle N^*, \tilde{E} \cup \tilde{E}^t\rangle\},$$

where $\tilde{E}^t$ is the edge set constructed up to stage t in the algorithm 3.1.3 applied to $\widetilde{\mathcal{M}}$. Note that $\tilde{t}_k = 0$ for all $k \in C_j$, the component of $\langle N^*, E\rangle$ connected to C_* by the edge e. Hence $\tilde{f}_k^t = 0$ for all stages t if $k \in C_j$. This means the players in C_j do not contribute to any edge. We now prove the lemma in three steps.

1. For $t \leq \tau - 1$, let $\tilde{e}^t = \{k, l\}$ and let $\tilde{C}^t = C_k^{t-1} \cup C_l^{t-1}$ be the component in the graph $\langle N^*, \tilde{E} \cup \tilde{E}^t\rangle$ formed by the addition of $\tilde{e}^t$. Then $\sum_{m\in\tilde{C}^t} \tilde{f}_m^t = 1$. To prove this, we distinguish several cases :

   - Suppose $\tilde{e}^t$ is not incident to $\tilde{C}_*^{t-1}$, the component of $*$ in $\langle N^*, \tilde{E} \cup \tilde{E}^t\rangle$. Then $\tilde{t}_m = \tilde{t}_k > t$ for all $m \in \tilde{C}^t$, hence

     - if $t < s$ then $\tilde{f}_m^t = f_m^t$ for $m \in \tilde{C}^t$. Moreover, $\tilde{C}^t = C^t$, the component in the graph $\langle N^*, E \cup E^t\rangle$ formed by the addition of $\tilde{e}^t = e^t$. Hence,

       $$\sum_{m\in\tilde{C}^t} \tilde{f}_m^t = \sum_{m\in C^t} f_m^t = 1$$

       by the assumption on $\mathcal{F}$.
     - if $t > s$ then $\tilde{f}_m^t = f_k^{t+1}$ for $m \in \tilde{C}^t$. Moreover, $\tilde{e}^t = e^{t+1}$ and $\tilde{C}^t = C^{t+1}$, the component in the graph $\langle N^*, E \cup E^{t+1}\rangle$ formed by the addition of e^{t+1}. Hence,

       $$\sum_{m\in\tilde{C}^t} \tilde{f}_m^t = \sum_{m\in C^{t+1}} f_m^{t+1} = 1$$

       by the assumption on $\mathcal{F}$.

   - Suppose that $\tilde{e}^t$ is incident to $\tilde{C}_*^{t-1}$. Then one of k and l, say k, lies in $\tilde{C}_*^{t-1}$. This means $\tilde{t}_m = \tilde{t}_l = t$ for all $m \in \tilde{C}_l^{t-1}$ and $\tilde{t}_m < t$ for all $m \in \tilde{C}_k^{t-1}$. Hence,

     - if $t < s$, then $\tilde{f}_m^t = \sum_{t'=t}^{\tau} f_m^{t'}$ for $m \in \tilde{C}_l^{t-1}$ and $\tilde{f}_m^t = 0$ for $m \in \tilde{C}_k^{t-1}$. This implies

       $$\begin{aligned}\sum_{m\in\tilde{C}^t} \tilde{f}_m^t &= \sum_{m\in\tilde{C}_l^{t-1}} \tilde{f}_m^t \\ &= \sum_{m\in\tilde{C}_l^{t-1}} \sum_{t'=t}^{\tau} f_m^{t'} \\ &= \sum_{m\in\tilde{C}_l^{t-1}} (\sum_{t'=1}^{\tau} f_m^{t'} - \sum_{t'=1}^{t-1} f_m^{t'}).\end{aligned}$$

Now $\tilde{C}_l^{t-1} = C_l^{t-1}$ is a union of a number, say p, of components $C_1, \ldots, C_p$ of the graph $\langle N^*, E \rangle$. Remember that for all $q \in \{1, \ldots, p\}$:

$$\sum_{m \in C_q} \sum_{t'=1}^{\tau} f_m^{t'} = 1,$$

hence

$$\sum_{m \in \tilde{C}_l^{t-1}} \sum_{t'=1}^{\tau} f_m^{t'} = \sum_{q=1}^{p} \sum_{m \in C_q} \sum_{t'=1}^{\tau} f_m^{t'} = p \tag{3.5.2}$$

and as $C_l^{t-1} = \bigcup_{q=1}^{p} C_q$ contains exactly those players that contributed to the $p-1$ edges in $\{e^1, \ldots, e^{t-1}\}$ that connect the components $(C_q)_{q=1}^{p}$ into C_l^{t-1},

$$\sum_{m \in \tilde{C}_l^{t-1}} \sum_{t'=1}^{t-1} f_m^{t'} = p - 1. \tag{3.5.3}$$

Equations (3.5.2) and (3.5.3) imply

$$\sum_{m \in \tilde{C}^t} \tilde{f}_m^t = p - (p-1) = 1.$$

– if $t \geq s$, then $\tilde{f}_m^t = f_m^{t+1}$, $\tilde{e}^t = e^{t+1}$ and $\tilde{C}^t = C^{t+1}$, the component in the graph $\langle N^*, E \cup E^{t+1} \rangle$ formed by the addition of e^{t+1}. Hence,

$$\sum_{m \in \tilde{C}^t} f_m^t = \sum_{m \in C^{t+1}} f_m^{t+1} = 1$$

by the assumption on $\mathcal{F}$.

2. For each component $C \in (N^*/\tilde{E})$ that does not contain the source : C is also a component of the graph $\langle N^*, E \rangle$ (because $\langle N^*, E \rangle$ and $\langle N^*, \tilde{E} \rangle$ differ only in the component of the source). Moreover, $\tilde{t}_k = \tilde{t}_l$ for all $k, l \in C$ and

   - if $\tilde{t}_k < s$ for all $k \in C$, then

$$\begin{aligned} \sum_{k \in C} \sum_{t=1}^{\tau-1} \tilde{f}_k^t &= \sum_{k \in C} \sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t \\ &= \sum_{k \in C} \left(\sum_{t=1}^{\tilde{t}_k - 1} \tilde{f}_k^t + \tilde{f}_k^{\tilde{t}_k} \right) \\ &= \sum_{k \in C} \left(\sum_{t=1}^{\tilde{t}_k - 1} f_k^t + \sum_{t=\tilde{t}_k}^{\tau} f_k^t \right) \\ &= \sum_{k \in C} \sum_{t=1}^{\tau} f_k^t \\ &= 1. \end{aligned}$$

   The first equality follows because $\tilde{f}_k^t = 0$ for $t > \tilde{t}_k$, the third equality because of the definition of $\tilde{f}_k^{\tilde{t}_k}$, and the fifth by the assumptions on $\mathcal{F}$.

- if $\tilde{t}_k \geq s$ for $k \in C$, then C is not connected to $*$ in $\langle N^*, \tilde{E} \cup \tilde{E}^{s-1}\rangle = \langle N^*, E \cup E^s\rangle$. Hence, according to $\mathcal{F}$, nobody in C contributes to e^s, which is an edge incident to C_*^s. This implies $f_k^s = 0$ for all $k \in C$. But then

$$\begin{aligned}
\sum_{k\in C}\sum_{t=1}^{\tau}\tilde{f}_k^t &= \sum_{k\in C}\sum_{t=1}^{\tilde{t}_k}\tilde{f}_k^t \\
&= \sum_{k\in C}\left(\sum_{t=1}^{s-1} f_k^t + \sum_{t=s}^{\tilde{t}_k} f_k^{t+1}\right) \\
&= \sum_{k\in C}\sum_{t=1}^{\tilde{t}_k+1} f_k^t \\
&= 1.
\end{aligned}$$

3. Furthermore, $\tilde{t}_k \leq t$ if $t \leq \tau-1$ and $k \in \tilde{C}_*^{t-1}$, the component of $*$ in $\langle N^*, \tilde{E}\cup\tilde{E}^{t-1}\rangle$. Hence, $\tilde{f}_k^t = 0$ by definition.

Steps 1, 2 and 3 imply $\tilde{\mathcal{F}} \in V^{\tilde{\mathcal{E}}}(\widetilde{\mathcal{M}})$. □

Corollary 3.5.4 Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be as above, let $\mathcal{E}$ be a sequence of choices made by algorithm 3.1.3 applied to $\mathcal{M}$, and let $\mathcal{F}$ be valid for $\mathcal{E}$. Then

$$x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) \geq x_k^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}}(\widetilde{\mathcal{M}}) \qquad \text{for all } k \in N,$$

where $\tilde{\mathcal{E}}$ is as defined in lemma 3.5.2 and $\tilde{\mathcal{F}}$ in lemma 3.5.3.

Proof : Knowing $x_k^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}}(\widetilde{\mathcal{M}}) = \sum_{t=1}^{\tilde{t}_k}\tilde{f}_k^t w(\tilde{e}^t)$, we distinguish three cases.

- If $\tilde{t}_k = 0$ then

$$\sum_{t=1}^{\tilde{t}_k}\tilde{f}_k^t w(\tilde{e}^t) = 0 \leq x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M})$$

- If $0 < \tilde{t}_k < s$ (with s as defined in lemma 3.5.2) then

$$\begin{aligned}
\sum_{t=1}^{\tilde{t}_k}\tilde{f}_k^t w(\tilde{e}^t) &= \sum_{t=1}^{\tilde{t}_k-1}\tilde{f}_k^t w(\tilde{e}^t) + \tilde{f}_k^{\tilde{t}_k} w(\tilde{e}^{\tilde{t}_k}) \\
&= \sum_{t=1}^{\tilde{t}_k-1} f_k^t w(e^t) + \left(\sum_{t=\tilde{t}_k}^{\tau} f_k^t\right) w(e^{\tilde{t}_k}) \\
&\leq \sum_{t=1}^{\tilde{t}_k-1} f_k^t w(e^t) + \sum_{t=\tilde{t}_k}^{\tau} f_k^t w(e^t) \\
&= \sum_{t=1}^{\tau} f_k^t w(e^t) \\
&= x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).
\end{aligned}$$

The second equation follows from the definition of $\tilde{\mathcal{F}}$ and the inequality holds because $\mathcal{E}$ is ordered by non-decreasing weights.

- If $\tilde{t}_k \geq s$ then k is not contained in $\tilde{C}_*^{s-1} = C_*^s$, so k is not allowed to contribute to e^s, i.e. $f_k^s = 0$. Hence,

$$\begin{aligned}
\sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t w(\tilde{e}^t) &= \sum_{t=1}^{s-1} f_k^t w(e^t) + \sum_{t=s}^{\tilde{t}_k} f_k^{t+1} w(e^{t+1}) \\
&= \sum_{t=1}^{s-1} f_k^t w(e^t) + \sum_{t=s+1}^{\tau} f_k^t w(e^t) \\
&= \sum_{t=1}^{\tau} f_k^t w(e^t) \\
&= x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).
\end{aligned}$$

As is all three cases the required inequality holds, the lemma is proven. □

This now enables us to prove theorem 3.1.8.

Theorem 3.1.8 For any mcse problem $\mathcal{M}$, for any sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 3.1.3 applied to $\mathcal{M}$ and any sequence of fraction vectors $\mathcal{F}$ that is valid for $\mathcal{E}$ the allocation $x^{\mathcal{E},\mathcal{F}}$, as defined in equation (3.1.1), is a core-allocation of the mcse game $(N, c^{\mathcal{M}})$ associated with $\mathcal{M}$.

Proof : Take any coalition $S \subseteq N$. We have to prove $\sum_{i \in S} x_i \leq c(S)$. Construct for coalition S a minimum-cost extension E^S containing only edges between components of $\langle N^*, E\rangle$ containing members of S^*, such that S is connected to the source in $\langle N^*, E \cup E^S\rangle$. Define $\mathcal{M}^S := \langle N, *, w, E \cup E^S\rangle$. Let p be one less than the number of components of $\langle N^*, E\rangle$ containing members of S^*. Then $|E^S| = p$ and the only difference between $\langle N^*, E\rangle$ and $\langle N^*, E \cup E^S\rangle$ is that the component C_*^S of the source in $\langle N^*, E \cup E^S\rangle$ is a union of the component C_* of the source and other components of $\langle N^*, E\rangle$. Construct the nested sequence $E = E_0 \subset \cdots \subset E_p = E \cup E^S$ such that $E_q \setminus E_{q-1}$ consists of exactly one edge which connects the component of the source $*$ in $\langle N^*, E_{q-1}\rangle$ to another component of $\langle N^*, E_{q-1}\rangle$. Consider the mcse problems $\mathcal{M}_q = \langle N^*, *, w, E_q\rangle$, where q varies from 0 to p, and note that for any $q > 0$, lemmata 3.5.2 and 3.5.3 and corollary 3.5.4 are applicable to the pair $\mathcal{M}_{q-1}$ and $\mathcal{M}_q = \widetilde{\mathcal{M}}_{q-1}$. Define $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{F}_0 = \mathcal{F}$ and for $1 \leq q \leq p$, define $\mathcal{E}_q = \widetilde{\mathcal{E}}_{q-1}$ and $\mathcal{F}_q = \widetilde{\mathcal{F}}_{q-1}$ recursively given $\mathcal{E}_{q-1}$ and $\mathcal{F}_{q-1}$, as in lemmata 3.5.2 and 3.5.3. Then by corollary 3.5.4,

$$x_k^{\mathcal{E}_q,\mathcal{F}_q}(\mathcal{M}_q) \leq x_k^{\mathcal{E}_{q-1},\mathcal{F}_{q-1}}(\mathcal{M}_{q-1})$$

for all $k \in N$ and all $1 \leq q \leq p$. Summing over $q \in \{1, \ldots, p\}$ and noticing that $\mathcal{M}_p = \mathcal{M}^S$ and $\mathcal{M}_0 = \mathcal{M}$, we obtain

$$x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S) = x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}_p) \leq x_k^{\mathcal{E}_0,\mathcal{F}_0}(\mathcal{M}_0) = x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M})$$

for all $k \in N$. Summing over $k \in N \setminus S$ yields

$$\sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S) \leq \sum_{k \in N \setminus S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \tag{3.5.4}$$

Denoting $(e_p^1, \ldots, e_p^{\tau-p}) := \mathcal{E}_p$, we see that the graph $\langle N^*, E \cup E^S \cup \{e_p^1, \ldots, e_p^{\tau-p}\}\rangle$ is a spanning extension for $\mathcal{M}$. On the other hand, $\langle N^*, E \cup \{e^1, \ldots, e^\tau\}\rangle$ is a minimum-cost spanning extension for $\mathcal{M}$, hence

$$\sum_{e \in E^S \cup \{e_p^1, \ldots, e_p^{\tau-p}\}} w(e) \geq \sum_{e \in \{e^1, \ldots, e^\tau\}} w(e) = c(N) = \sum_{k \in N} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \tag{3.5.5}$$

Now by definition of E^S,

$$\sum_{e \in E^S} w(e) = c(S) \tag{3.5.6}$$

and by efficiency of $x^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S)$ (cf. lemma 3.1.4),

$$\sum_{e \in \{e_p^1, \ldots, e_p^{\tau-p}\}} w(e) = \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S). \tag{3.5.7}$$

Plugging equations (3.5.6) and (3.5.7) into inequality 3.5.5 and using inequality 3.5.4, we obtain

$$\begin{aligned} c(S) + \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S) &\geq \sum_{k \in N} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) \\ &= \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) + \sum_{k \in N \setminus S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \\ &\geq \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) + \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}^S), \end{aligned} \tag{3.5.8}$$

which is equivalent to

$$c(S) \geq \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).$$

As we proved in lemma 3.1.4 that $x^{\mathcal{E},\mathcal{F}}(\mathcal{M})$ is efficient, it is a core element of $(N, c^{\mathcal{M}})$.
□

We now give a proof of proposition 3.1.10.

Proposition 3.1.10 For any mcse problem $\mathcal{M}$, for any $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ constructed by the algorithm 3.1.3 applied to $\mathcal{M}$,

$$D^{\mathcal{E}}(\mathcal{M}) = D^{\widetilde{\mathcal{E}}}(\mathcal{M}).$$

Proof : First we prove that $D^{\mathcal{E}}$ is independent of the order in which a particular mcse is constructed. Suppose that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are two sequences constructed in algorithm 3.1.3 applied to $\mathcal{M}$, both leading to the same mcse E', i.e. $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ differ only in their order. Because in algorithm 3.1.3, the edges in $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are ordered by non-decreasing cost, this means that $\widetilde{\mathcal{E}}$ equals $\mathcal{E}$ except for edges of the same cost that are swapped. If more than two edges are swapped, it is possible to construct a series $\mathcal{E} = \mathcal{E}^0, \ldots, \mathcal{E}^p = \widetilde{\mathcal{E}}$ of sequences all leading to the same mcse E', with for any $q \leq p$, $\mathcal{E}^q$ equal to $\mathcal{E}^{q-1}$ except for exactly two *subsequent* edges with the same cost that are swapped.

So it suffices to prove $D^{\mathcal{E}} = D^{\widetilde{\mathcal{E}}}$ with $\mathcal{E}$ equal to the sequence $(e^1, \ldots, e^t, e^{t+1}, \ldots, e^\tau)$ and $\widetilde{\mathcal{E}}$ equal to the sequence $(e^1, \ldots, e^{t+1}, e^t, \ldots, e^\tau)$, for some $t < \tau$ with $w(e^t) = w(e^{t+1})$. Two cases will be distinguished :

1. the components C^t and C^{t+1} formed by adjoining the edges e^t and e^{t+1} to $\langle N^*, E \cup \{e^1, \ldots, e^{t-1}\}\rangle$ and $\langle N^*, E \cup \{e^1, \ldots, e^t\}\rangle$, respectively are disjoint. For $\mathcal{F} \in V^{\mathcal{E}}$, define $\widetilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^\tau)$ by

$$\begin{array}{rcll} \tilde{f}^s & = & f^s & \text{if } s \notin \{t, t+1\}, \\ \tilde{f}^t & = & f^{t+1}, & \\ \tilde{f}^{t+1} & = & f^t. & \end{array}$$

   Obviously, $\widetilde{\mathcal{F}} \in V^{\widetilde{\mathcal{E}}}$ and $x^{\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}}(\mathcal{M}) = x^{\mathcal{E}, \mathcal{F}}(\mathcal{M})$.

2. the components C^t and C^{t+1} are not disjoint. Then we are in the situation drawn in figure 3.4 : C^t consists of two components C_1 and C_2, connected by e^t, and C^{t+1} is formed by connecting C_3 to C^t via e^{t+1}. Without loss of generality, we suppose e^{t+1} is incident to C_2.

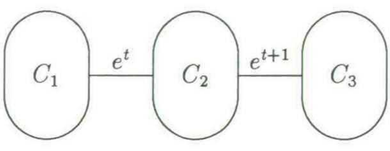


Figure 3.4: e^t links C_1 to C_2 and e^{t+1} links $C_1 \cup C_2$ to C_3.

Now let $\mathcal{F} \equiv (f^1, \ldots, f^\tau) \in V^{\mathcal{E}}$ be a valid sequence of fraction vectors, and define $\widetilde{\mathcal{F}} \equiv (\tilde{f}^s)_{s=1}^{\tau}$ by

$$\begin{array}{rcll} \tilde{f}^s_k & = & f^s_k & \text{if } s \notin \{t, t+1\} \text{ or } k \notin C_1 \cup C_2 \cup C_3, \\ \tilde{f}^t_k & = & \left\{ \begin{array}{ll} 0 & \text{if } k \in C_1, \\ f^{t+1}_k + g_k & \text{if } k \in C_2, \\ f^{t+1} & \text{if } k \in C_3, \end{array} \right. & \\ \tilde{f}^{t+1}_k & = & \left\{ \begin{array}{ll} f^t_k + f^{t+1}_k & \text{if } k \in C_1, \\ f^t_k - g_k & \text{if } k \in C_2, \\ 0 & \text{if } k \in C_3, \end{array} \right. & \end{array}$$

where $g \in \mathbf{R}^{C_2}$ is a vector such that

$$\left\{ \begin{array}{l} \sum\limits_{k \in C_2} g_k = \sum\limits_{k \in C_1} f^{t+1}_k \\ f^t_k \geq g_k \geq 0 \text{ for all } k \in C_2. \end{array} \right. \tag{3.5.9}$$

The system of equations (3.5.9) are feasible, because $\mathcal{F} \in V^{\mathcal{E}}$ implies

$$\sum_{k\in C_2} f_k^t = 1 - \sum_{k\in C_1} f_k^t \geq \sum_{k\in C_1} f_k^{t+1} = \sum_{k\in C_2} g_k.$$

One easily sees that $\widetilde{\mathcal{F}}$ is valid for $\widetilde{\mathcal{E}}$ and that

$$x_k^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}} = \sum_{t=1}^{\tau} \tilde{f}_k^t w(\tilde{e}^t) = \sum_{t=1}^{\tau} f_k^t w(e^t) = x_k^{\mathcal{E},\mathcal{F}}.$$

Hence $D^{\mathcal{E}}$ is independent of the order of $\mathcal{E}$, so we define

$$D^{E'} := D^{\mathcal{E}}$$

for any sequence $\mathcal{E}$ constructed by the algorithm 3.1.3 that leads to E'.

We still have to prove that $D^{E'} = D^{\tilde{E}}$ for all mcses E' and $\tilde{E}$ of $\mathcal{M}$. First, we need to know more about the structure of the set of all mcses of $\mathcal{M}$. Now constructing an mcse for the mcse problem $\mathcal{M}$ is equivalent to constructing an mcst on the associated mcst problem $\langle N_E, *_E, w_E\rangle$ (cf. definition 3.2.1). It is well known that for any two mcsts $\langle N^*, T\rangle$ and $\langle N^*, \tilde{T}\rangle$ of an mcst problem $\langle N, *, w\rangle$, for every edge $e \in T \setminus \tilde{T}$, there exists an edge $\tilde{e} \in \tilde{T} \setminus T$ such that $\langle N^*, T \cup \{\tilde{e}\} \setminus \{e\}\rangle$ is again a minimum-cost spanning tree.

Suppose that $\langle N^*, E \cup E'\rangle$ and $\langle N^*, E \cup \tilde{E}\rangle$ are two mcses for $\mathcal{M}$. Then with E'_E and $\tilde{E}_E$ as defined in equation (3.2.2), note that $\langle N^*_E, E'_E\rangle$ and $\langle N^*_E, \tilde{E}_E\rangle$ are mcsts of $\langle N_E, *_E, w_E\rangle$.

Now for every $e' \in E' \setminus \tilde{E}$ and e'_E defined as in equation (3.2.1), it holds that either $e'_E \in E'_E \setminus \tilde{E}_E$ or $e'_E \in E'_E \cap \tilde{E}_E$. In the former case, there exists an edge $\{C, D\} \in \tilde{E}_E \setminus E'_E$ such that

$$\langle N^*_E, E'_E \cup \{\{C, D\}\} \setminus \{e'_E\}\rangle$$

is also a minimum-cost spanning tree. By definition of $\tilde{E}_E$ there exists an edge $\tilde{e} \in \tilde{E} \setminus E'$ with $\tilde{e}_E = \{C, D\}$ and $w(\tilde{e}) = w_E(e) = w_E(e'_E) = w(e')$. In the latter case, there exists an edge $\tilde{e} \in \tilde{E} \setminus E'$ with $\tilde{e}_E = e'_E$, so $\tilde{e}$ connects the same components as e'. Because both $\tilde{E}$ and E' are mcses, $w(\tilde{e}) = w(e')$. So in both cases, we obtain that

$$E \cup E' \cup \{\tilde{e}_E\} \setminus \{e'_E\}$$

is a minimum-cost spanning extension of $\mathcal{M}$, which differs one edge less from $\tilde{E}$ than E' does.

Hence, to prove that $D^{E'}$ is independent of the mcse E', it suffices to prove that $D^{E'} = D^{\tilde{E}}$ for two mcses E' and $\tilde{E}$ of $\mathcal{M}$ with $|E' \setminus \tilde{E}| = 1$.

Because E' and $\tilde{E}$ are minimum-cost extensions, the edge $e' \in E' \setminus \tilde{E}$ and the edge $\tilde{e} \in \tilde{E} \setminus E'$ have to have the same cost. Now order E' and $\tilde{E}$ by non-decreasing cost into sequences $\mathcal{E} = (e^1, \ldots, e^{s-1}, e', e^{s+1}, \ldots, e^\tau)$ and $\widetilde{\mathcal{E}} = (e^1, \ldots, e^{s-1}, \tilde{e}, e^{s+1}, \ldots, e^\tau)$

where s equals the number of edges in E' with cost not greater than $w(e') = w(\tilde{e})$. Then $w(e^t) > w(e')$ for all $t > s$, and moreover $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are two sequences that can be constructed by algorithm 3.1.3 applied to $\mathcal{M}$.

Consider the graph $\langle N^*, E \cup \{e^1, \ldots, e^{s-1}, e'\}\rangle$. As the next edge e^{s+1} has greater cost than $\tilde{e}$, it has to be the case that adding $\tilde{e}$ would introduce a cycle. But adding

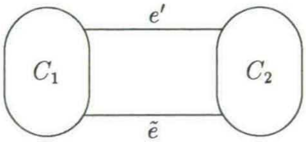


Figure 3.5: e' and $\tilde{e}$ both link C_1 to C_2.

$\tilde{e}$ to $\langle N^*, E \cup \{e^1, \ldots, e^{s-1}\}\rangle$ does not introduce a cycle. This means that e' and $\tilde{e}$ connect the same two components of $\langle N^*, E \cup \{e^1, \ldots, e^{s-1}\}\rangle$ (see figure 3.5). Hence the components of the graphs $\langle N^*, E \cup E^t\rangle$ and $\langle N^*, E \cup \tilde{E}^t\rangle$ coincide for all t, which implies that $V^{\mathcal{E}} = V^{\widetilde{\mathcal{E}}}$. Together with $w(e') = w(\tilde{e})$, this implies $D^{E'} = D^{\mathcal{E}} = D^{\widetilde{\mathcal{E}}} = D^{\tilde{E}}$. $\square$

Chapter 4

More on mcse problems

This chapter, which is based on Feltkamp, Tijs and Muto (1994c), proposes two cost-allocation rules for minimum-cost spanning extension problems : the proportional rule and the decentralized rule. Both rules specify not only allocations of the cost of an mcse, but also *which* extension of the network should be constructed. It is shown that the cost of an edge can be allocated as soon as this edge is constructed. Section 4.3 characterizes the proportional rule axiomatically, using among other things, efficiency and maximality.

Section 3.3 showed that if an mcse is constructed according to algorithm 3.1.3 and the cost of every edge is shared among the players such that at every stage the remaining obligations of the players in one component are equal, the resulting allocation is the equal-remaining-obligations allocation of the mcse problem $\mathcal{M}$.

This chapter will assume instead that at every stage, the cost of an edge is shared proportionally to the remaining obligations of the players in the components constructing this edge. This yields the proportional allocation if the mcse is constructed according to the generalized Kruskal algorithm (algorithm 3.1.3) and the decentralized allocation if a decentralized algorithm is used.

4.1 The proportional rule

The *proportional rule* is constructed by the following algorithm : construct the edges of a minimum-cost spanning extension as in Kruskal's algorithm. Each time an edge is constructed, its cost is divided proportionally to the (remaining) obligations, among the players in the components being linked. More precisely :

Algorithm 4.1.1 (the proportional rule)
input : an mcse problem $\mathcal{M}$
output : a sequence $\mathcal{E}$ of edges leading to an mcse and a cost allocation $\mathrm{PRO}^{\mathcal{E}}(\mathcal{M})$

1. Given $\mathcal{M} \equiv \langle N, *, w, E\rangle$, define

$$\begin{array}{rcll} t & = & 0 & \text{the initial stage,} \\ \tau & = & |N^*/E| - 1 & \text{the number of stages,} \\ E^0 & = & E & \text{the initial edge set,} \\ o_i^0 & = & o_i & \text{the initial obligation (cf. equation 3.3.1) for all } i \in N. \end{array}$$

2. While $t < \tau$, do steps 3 to 7.

3. $t := t + 1$.

4. At stage t, given E^{t-1}, choose one edge $e^t \in E_{N^*} \setminus E^{t-1}$ among the cheapest edges such that the graph $\langle N^*, E^{t-1} \cup \{e^t\}\rangle$ does not contain more cycles than $\langle N^*, E^{t-1}\rangle$.

5. If C^t is the connected component just formed by adding the edge e^t to the graph $\langle N^*, E^{t-1}\rangle$, define the vector $f^t = (f_i^t)_{i \in N}$ of fractions the players contribute by

$$f_i^t = \begin{cases} \dfrac{o_i^{t-1}}{\sum\limits_{l \in C^t} o_l^{t-1}} & \text{if } i \in C^t, \\ 0 & \text{if } i \notin C^t. \end{cases}$$

6. Define the *remaining obligation* after stage t by $o_k^t := o_k^{t-1} - f_k^t$ for all $k \in N$.

7. Define $E^t := E^{t-1} \cup \{e^t\}$.

8. Define $\mathcal{E} = (e^1, \ldots, e^\tau)$.

9. Define $\mathrm{PRO}^{\mathcal{E}}(\mathcal{M}) := \sum\limits_{t=1}^{\tau} f^t w(e^t)$.

Note that at every stage, the sum of the obligations of the players in a component that does not contain the source equals 1, and the obligation of any player in the component of the source equals 0. Hence, in step 5, the denominator equals 1 or 2, depending on whether C^t contains the source or not. Note that it is never zero.

As the allocation generated by this algorithm depends on the choices of edges made, define the proportional rule (or solution) by

$$\mathrm{PRO}(\mathcal{M}) := \{(\mathcal{E}, \mathrm{PRO}^{\mathcal{E}}(\mathcal{M})) \mid \mathcal{E} \text{ is obtained by the algorithm 4.1.1}\}.$$

Note that this algorithm constructs exactly one sequence of fraction vectors per sequence of edges chosen. As only finitely many minimum-cost spanning extensions exist, $\mathrm{PRO}(\mathcal{M})$ is finite for any mcse problem $\mathcal{M}$.

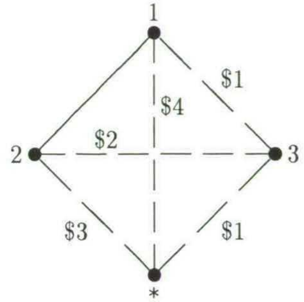


Figure 4.1: The edge $\{1,2\}$ is initially present.

Example 4.1.2 In the mcse game associated with the graph depicted in figure 4.1, coalition $\{1\}$ can link itself to the root using player 2, but $\{1\}$ cannot use player 3. Hence, $c(\{1\}) = 3$. Similarly, the costs for other coalitions are : $c(\{2\}) = 3$, $c(\{3\}) = 1$, $c(\{1,2\}) = 3$, $c(\{1,3\}) = c(\{2,3\}) = c(\{1,2,3\}) = 2$.

Applying the algorithm to this problem, note that $o^0 = (0.5, 0.5, 1)$. A possible first edge is edge $\{*, 3\}$. Then $f^1 = (0,0,1)$ and the remaining obligation of player three is zero, while the obligations of the others remain unchanged. The next edge has to be $\{1,3\}$, which implies $f^2 = (0.5, 0.5, 0)$. Hence, the (final) allocation equals

$$1(0.5, 0.5, 0) + 1(0, 0, 1) = (0.5, 0.5, 1).$$

The only other possible first edge is $\{1,3\}$, yielding $f^1 = (0.5, 0.5, 1)/2$ and $o^2 = (0.5, 0.5, 1) - (0.5, 0.5, 1)/2 = (0.25, 0.25, 0.5)$. Then $\{*, 3\}$ is the second edge, yielding $f^2 = (0.25, 0.25, 0.5)/1$. Hence, the allocation is $1(0.25, 0.25, 0.5) + 1(0.25, 0.25, 0.5) = (0.5, 0.5, 1)$. The two sequences thus yield the same allocation.

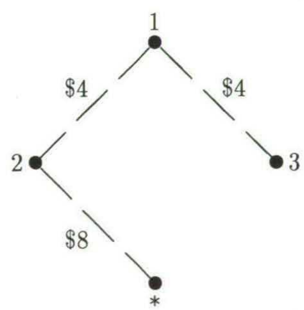


Figure 4.2: No edge is initially present and edges not drawn cost \$100.

The following example shows that the proportional rule can yield more than one allocation.

Example 4.1.3 In the problem depicted in figure 4.2, if the edge $\{1,2\}$ is constructed first, then the proportional allocation is $(5,5,6)$; if the edge $\{1,3\}$ is constructed first, the allocation is $(5,6,5)$.

Chapter 3 defined the irreducible core IC($\mathcal{M}$) of an mcse problem $\mathcal{M}$ and generated it by associating with every sequence of edges constructed as in algorithm 4.1.1, a set of *valid* sequences of fraction vectors. For a sequence $(e^1,\ldots,e^\tau)$ of edges, valid sequences of fraction vectors are those sequences $(f^1,\ldots,f^\tau)$ that satisfy the following :

- Each component of the original graph that does not contain the source has to pay fractions of edges that total 1.
- At each stage, the players in the component that contains the source do not contribute to the cost of the edge constructed.
- At each stage, the cost of the edge that is constructed is shared by the players in the two components that it joins.

Moreover, by proposition 3.2.13, if $\mathcal{E}=(e^1,\ldots,e^\tau)$ is a sequence of edges of an mcse of the mcse problem $\mathcal{M}$ generated by algorithm 3.1.3 and $\mathcal{F}=(f^1,\ldots,f^\tau)$ is valid for $\mathcal{E}$, then the vector

$$x^{\mathcal{E},\mathcal{F}} := \sum_{t=1}^{\tau} f^t w(e^t)$$

lies in IC($\mathcal{M}$). It is straightforward to see that for an mcse problem $\mathcal{M}$, the proportional algorithm generates the same sequence of edges $\mathcal{E}$ as algorithm 3.1.3 and a valid sequence $\mathcal{F}$ of fraction vectors. Hence, $\mathrm{PRO}^{\mathcal{E}}(\mathcal{M}) = x^{\mathcal{E},\mathcal{F}} \in \mathrm{IC}(\mathcal{M})$. The set of allocations generated by the proportional rule is thus a refinement of the irreducible core, which is a subset of the core. In particular, the allocations generated by the proportional rule are all core elements of the mcse game. This proves (once more) that mcse games are balanced.

4.2 The decentralized rule

Prim and Dijkstra's algorithm, algorithm 3.1.3, the ERO algorithm, and the proportional algorithm, are centralized algorithms, in the sense that one edge is constructed per stage. (Note that the last three algorithms construct the same sequences of edges.) However, a situation might occur in which at each stage in the construction, every component which does not contain the source greedily starts to build the cheapest edge that links it to another component.

The idea of building a minimum-cost spanning tree in this way dates back in its first documented full formulation to Borůvka (1926a, 1926b). He considered minimum-cost spanning tree problems, but the distinction is minimal if the only goal is to construct a network. It is only if one wants to allocate costs that the difference is essential. This so-called decentralized algorithm builds a network in fewer stages than all previously described (centralized) algorithms, though the stages themselves are larger.

We associate a cost allocation with this algorithm that is similar to the proportional rule : at every stage, each component that builds an edge has to pay for it, unless two components want to build the same edge, in which case each pays half.

Algorithm 4.2.1 (the decentralized rule)
input : an mcse problem $\mathcal{M}$
output : an extension E' and a cost allocation $\mathrm{DEC}(\mathcal{M})$

1. Given $\mathcal{M} \equiv \langle N, *, w, E \rangle$, define

$$\begin{array}{rcll} t & = & 0 & \text{the initial stage,} \\ E^0 & = & E & \text{the initial edge set.} \\ o_i^0 & = & o_i & \text{the initial obligation for all } i \in N. \end{array}$$

2. While the graph $\langle N^*, E^t \rangle$ is not yet connected, do steps 3 to 7.

3. $t := t + 1$.

4. At stage t, each component C of $\langle N^*, E^{t-1} \rangle$ that does not contain the source chooses a cheapest edge e_C^t linking C to another component of $\langle N^*, E^{t-1} \rangle$.

5. Define the vector $f^t = (f_k^t)_{k \in N} \in \Delta^N$ of fractions by

$$f_k^t = \begin{cases} o_k^{t-1} & \text{if only } C_k^{t-1} \text{ chooses } e^t_{C_k^{t-1}} \\ o_k^{t-1}/2 & \text{if another component also chooses } e^t_{C_k^{t-1}} \\ 0 & \text{if } k \in C_*, \end{cases}$$

for all $k \in N$. As usual, C_k^{t-1} denotes the component containing k in the graph $\langle N^*, E^{t-1} \rangle$ constructed at stage $t - 1$.

6. Define the remaining obligation after stage t by $o_k^t := o_k^{t-1} - f_k^t$ for all $k \in N$.

7. Define $E^t := E^{t-1} \cup \{e_C^t \mid C \in N^*/E^{t-1} \text{ and } * \notin C\}$.

8. Denote the number of stages needed by s.

9. Define the extension $E' := E^t \setminus E$.

10. Define the decentralized allocation $\mathrm{DEC}(\mathcal{M})$ by

$$\mathrm{DEC}_k(\mathcal{M}) := \sum_{t=1}^{s} f_k^t w(e^t_{C_k^{t-1}})$$

for all $k \in N$.

Example 4.2.2 Consider the mcse problem of figure 4.1 and modify the cost of edge $\{1,3\}$ to \$1.5. Applying algorithm 4.2.1 to this mcse problem yields in the first stage : component $\{1,2\}$ constructs edge $\{1,3\}$ and component $\{3\}$ constructs edge $\{*,3\}$. The cost of edge $\{1,3\}$ is shared by players 1 and 2 proportionally to their initial obligations (see example 4.1.2), and player 3 pays the edge he chose, hence $f^1 = (0.5, 0.5, 1)$. Now after the construction of these edges, every player is connected to the source, hence $s := 1$ and the decentralized allocation $\text{DEC}(\mathcal{M}) = (0.75, 0.75, 1)$.

Algorithm 4.2.1 can generate a network $\langle N^*, E' \rangle$ containing cycles when applied to an arbitrary mcse problem, but on generic mcse problems, where all weights are different, it does not.

Definition 4.2.3 An mcse problem $\langle N, *, w, E \rangle$ is called *generic* if for every pair $e \neq \tilde{e}$ of edges,

$$w(e) \neq w(\tilde{e}).$$

Note that on the class of generic mcse problems, for each component $C \in N^*/E^t$, only one edge e^t_C exists that can be chosen in step 4; the decentralized rule thus constructs a unique mcse and allocation on this class of problems.

Theorem 4.2.4 If the mcse problem $\langle N, *, w, E \rangle$ is generic, the decentralized algorithm generates an mcse.

Proof : Let $\langle N, *, w, E \rangle$ be a generic mcse problem. Clearly, algorithm 4.2.1 leads to a connected graph. The only way that a cycle can be introduced in this graph is that after a stage $t-1$, there are $p \geq 3$ components $C_1, \ldots, C_p$, such that at stage t, for each $1 \leq q < p$, component C_q constructs an edge e_q connecting it to component C_{q+1}, while component C_p constructs an edge e_p connecting it to component C_1. Now because C_q prefers e_q to e_{q-1} for all $1 < q \leq p$ and C_1 prefers e_1 to e_p, it follows that

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_p) \geq w(e_1). \tag{4.2.1}$$

But this can only hold if all inequalities in 4.2.1 are equalities, which is impossible in a generic mcse problem.

Suppose the network $\langle N^*, E^s \rangle$ constructed by algorithm 4.2.1 is not an mcse of $\langle N^*, E \rangle$. Then there exists a minimum-cost spanning extension $\langle N^*, \tilde{E} \rangle$ that satisfies

$$\sum_{e \in \tilde{E}} w(e) < \sum_{e \in E^s} w(e).$$

Now consider the earliest stage t featuring a component C of $\langle N^*, E^{t-1} \rangle$ that constructs an edge e^t_C that is not present in $\tilde{E}$. This means that all edges in $\langle N^*, E^{t-1} \rangle$ are part of $\tilde{E}$. Adding e^t_C to $\tilde{E}$ introduces a cycle, which has to include another edge $\tilde{e}$ linking C to another component of $\langle N^*, E^{t-1} \rangle$ because e^t_C has only one end-point in C. Now e^t_C

was the cheapest edge linking C to another component of $\langle N^*, E^{t-1}\rangle$, so $w(\tilde{e}) > w(e^t_C)$, and deleting $\tilde{e}$ from $\tilde{E} \cup \{e^t_C\}$ produces a spanning extension that has smaller cost than $\langle N^*, \tilde{E}\rangle$. This is a contradiction; the algorithm hence produces an mcse. □

The next theorem states that applied to a generic mcse problem, the decentralized algorithm constructs core elements of the associated mcse game.

Theorem 4.2.5 On the class of generic mcse problems, the allocations generated by the decentralized algorithm are elements of the irreducible core.

Proof : To prove this, we only need to prove that the allocations generated by the decentralized algorithm can also be generated by a sequence of edges that can be generated by algorithm 4.1.1, together with fraction vectors that are valid for this sequence, as defined in section 4.1.

Let $\mathcal{M} \equiv \langle N, *, w, E\rangle$ be a generic mcse problem. Construct the sequence $\tilde{\mathcal{E}}$ of edges as follows : at each stage t, order the edges constructed at stage t by the decentralized algorithm by increasing cost into a sequence $\mathcal{E}^t$. Then concatenate these sequences, obtaining a sequence

$$\tilde{\mathcal{E}} \equiv (\tilde{e}^1, \ldots, \tilde{e}^\tau)$$

where $\tau = |N^*/E| - 1$ is the number of edges in an mcse.

Construct the sequence $\tilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^\tau)$ of fraction vectors by

$$\tilde{f}^u_k := \begin{cases} f^t_k & \text{if } \tilde{e}^u = e^t_{C^{t-1}_k}, \\ 0 & \text{otherwise.} \end{cases}$$

One easily sees that $\tilde{\mathcal{F}}$ is valid for $\tilde{\mathcal{E}}$ and that

$$\mathrm{DEC}(\mathcal{M}) = x^{\tilde{\mathcal{E}}, \tilde{\mathcal{F}}}. \tag{4.2.2}$$

If the sequence $\tilde{\mathcal{E}}$ were ordered by increasing cost, we could conclude that the decentralized rule generates an element of the irreducible core by invoking theorem 3.2.13. Because in general $\tilde{\mathcal{E}}$ will not be ordered by increasing cost, we will first investigate the structure of $\tilde{\mathcal{E}}$ and then construct an ordered sequence.

In any component C^t of $\langle N^*, E^t\rangle$ that does not contain the source and that is constructed at stage t by the decentralized algorithm, there is exactly one edge e^t that is constructed by *two components* (call these C_0 and C_1) of $\langle N^*, E^{t-1}\rangle$, all other edges are constructed by only *one of the two components* which they connect. Now this edge e^t is cheaper than any other edge $\tilde{e}^t_C$ that is constructed at stage t by any component C of $\langle N^*, E^{t-1}\rangle$ that is a subset of C^t. To see this, consider the 'path' formed at stage t by the edge constructed by C, the edge constructed by the component that C connects itself to, the edge constructed by the component that that component connects itself to, and so on until finally the edge e^t is reached. (If the edge e^t is not reached, the last component in the sequence chooses another edge than the preceding ones, and the path can be prolonged.) We denote the number of edges in this path by p and number the components and edges in this path, starting with $e_1 = e^t$ which links components C_0 and C_1 (see figure 4.3).

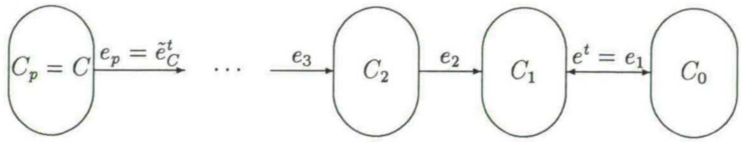


Figure 4.3: The 'path' between C_p and C_0.

For all $0 < q < p$, the component C_q prefers edge e_q to edge e_{q+1}. Hence,

$$w(e^t) = w(e_1) < w(e_2) < \ldots < w(e_p) = w(\tilde{e}^t). \tag{4.2.3}$$

We now prove that the edge $e^{t+1}_{C^t}$ chosen by C^t at stage $t+1$ is more expensive than all edges between $e^{t+1}_{C^t}$ and e^t. For any edge e that can be chosen by C^t in stage $t+1$, e could have been chosen at stage t by one of the components C^{t-1} of $\langle N^*, E^{t-1}\rangle$ that is a subset of C^t. Hence, because e was not chosen by C^{t-1}, it must have higher cost than $e^t_{C^{t-1}}$, the edge that C^{t-1} chose. Using equation 4.2.3, we obtain

$$w(e) > w(e^t_{C^{t-1}}) > w(e^t).$$

As this holds for all edges e that can be chosen by C^t, it also holds for the edge $e^{t+1}_{C^t}$ chosen by C^t at stage $t+1$. Hence an edge that is chosen by a component C at a stage t is more expensive than the edge chosen in the previous stage by *two components* of $\langle N^*, E^{t-1}\rangle$ that are subsets of the component C. Furthermore, all edges on the path from C^{t-1} to e^t are less expensive than $e^{t+1}_{C^t}$ and hence all edges between $e^{t+1}_{C^t}$ and e^t are cheaper than $e^{t+1}_{C^t}$. This implies that in the sequence $\mathcal{E}'$, obtained by sorting $\tilde{\mathcal{E}}$ according to increasing cost, the edges between $e^{t+1}_{C^t}$ and e^t appear before $e^{t+1}_{C^t}$, just like they did in the sequence $\mathcal{E}$.

According to the algorithm 4.2.1, a player only contributes to the cost of an edge if his remaining obligation is positive, which implies that all previous edges he paid were chosen by two components. Hence, the edges to which a player contributes according to the algorithm 4.2.1 are ordered by increasing cost in the sequence $\mathcal{E}$ and their relative order is the same in the sequence $\mathcal{E}'$.

Defining $\mathcal{F}'$ accordingly by

$$f'^r := \tilde{f}^u \qquad \text{for the unique } u \text{ such that } e'^r = \tilde{e}^u$$

for $r \in \{1, \ldots, \tau\}$, we see that

$$x^{\tilde{\mathcal{E}}, \tilde{\mathcal{F}}} = x^{\mathcal{E}', \mathcal{F}'}. \tag{4.2.4}$$

Furthermore, that $\tilde{\mathcal{F}}$ is valid for $\tilde{\mathcal{E}}$ implies $\mathcal{F}'$ is valid for $\mathcal{E}'$:

- Each component in the original graph which does not contain the source pays fractions of edges that total one, because the fractions have only been reordered.
- At each stage, the players in the component of the source do not contribute to the cost of the edge constructed, because in the sequence $\widetilde{\mathcal{F}}$, a player can only pay for an edge if all previous edges to which this player contributed were paid by the two components that they connect. As in the sequence $\widetilde{\mathcal{F}}$ the component of the source never pays for an edge, no player that joins the original component of the source can later on contribute to other edges : the edge paid when connecting to the component of the source was necessarily chosen by only one component. Players in the original component of the source do not pay for any edges according to $\widetilde{\mathcal{E}}$, hence they do not pay for any edge according to $\mathcal{E}'$ either.
- Whenever a player k pays an edge e^r and later pays an edge e^u, the edges that connect these edges in the mcse appear in $\mathcal{E}'$ before e^u is constructed. Hence, if k lies in the component incident to e^r, k also lies in the component incident to e^u. Using induction, we see that at each stage, the cost of the constructed edge is paid by players in the components it connects.

So, $\mathcal{E}'$ is a sequence of edges of an mcse ordered by increasing cost and $\mathcal{F}'$ is valid for $\mathcal{E}'$, hence $x^{\mathcal{E}',\mathcal{F}'} \in \mathrm{IC}(\mathcal{M})$. Using equations 4.2.2 and 4.2.4 we conclude that the decentralized value lies in the irreducible core of $\mathcal{M}$. □

4.3 An axiomatic characterization of the proportional rule

In sections 4.1 and 4.2, we introduced two rules for mcse problems. We axiomatically characterize the proportional rule in this section.

We define a *solution* of mcse problems as a function ψ assigning to every mcse problem $\langle N, *, w, E\rangle$, a set

$$\psi(\langle N,*,w,E\rangle) \subseteq \left\{ ((e^1,\dots,e^\tau),x) \;\middle|\; \begin{array}{l} \langle N^*, E\cup\{e^1,\dots,e^\tau\}\rangle \text{ is a connected} \\ \text{graph, the edges } e^1,\dots,e^\tau \text{ are ordered} \\ \text{by non-decreasing weight and } x\in\mathbf{R}^N \\ \text{satisfies } \sum_{i\in N} x_i \geq \sum_{t=1}^{\tau} w(e^t). \end{array} \right\}.$$

Note that in contrast with solutions of chapter 3, a solution here contains information about which sequence of edges is constructed. This is because the proportional rule depends on the sequence of edges constructed, unlike the ERO rule or the irreducible core. We enumerate desirable properties of a solution ψ of mcse problems. Some of these already appeared in chapter 3 in a slightly different form, due to the different definition of solution in that chapter.

Definition 4.3.1

Eff ψ is *efficient* if for all $((e^1,\ldots,e^\tau),x) \in \psi(\mathcal{M})$, for all $\mathcal{M}$, $E \cup \{e^1,\ldots,e^\tau\}$ is a minimum-cost spanning extension and

$$\sum_{i\in N} x_i = \sum_{t=1}^{\tau} w(e^t).$$

MC ψ has the *minimal contribution property* if in every mcse problem, every component that does not contain the source contributes at least the cost of a minimum-cost edge that connects two components of the graph $\langle N^*, E\rangle$. In formula : for all $\mathcal{M} \equiv \langle N, *, w, E\rangle$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, for each component $C \in N^*/E$ that does not contain the source,

$$\sum_{i\in C} x_i \geq \min\{w(e) \mid e \text{ connects two components of } \langle N^*, E\rangle\}.$$

FSC ψ has the *free-for-source-component* property if for all $\mathcal{M}$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, we have

$$x_i = 0$$

for all i in the component of the source in the graph $\langle N^*, E\rangle$.

ET ψ satisfies *equal treatment* if for all $\mathcal{M}$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, for all components $C \in N^*/E$, and for all players i and $j \in C$,

$$x_i = x_j.$$

Recall that the *edge-reduced mcse problem* $\mathcal{M}^e$ of an mcse problem $\mathcal{M} \equiv \langle N, *, w, E\rangle$ and an edge e that connects two components of $\langle N^*, E\rangle$, is defined by

$$\mathcal{M}^e = \langle N, *, w, E \cup \{e\}\rangle.$$

The next three properties of a solution ψ relate the solution of an mcse problem and the solutions of its edge-reduced mcse problems.

Definition 4.3.2

ES ψ satisfies *equal share* if for any $\mathcal{M}$, for all all $((e^1,\ldots,e^\tau),x) \in \psi(\mathcal{M})$ with e^1 connecting two components C_1 and C_2, there exists a $(\widetilde{\mathcal{E}}, \tilde{x}) \in \psi(\mathcal{M}^{e^1})$ such that

$$\sum_{i\in C_1} (x_i - \tilde{x}_i) = \sum_{i\in C_2} (x_i - \tilde{x}_i).$$

In effect, this property requires that the two components connected in the first step of a solution participate in equal amounts in the cost of the edge which connects them.

Loc ψ is *local* if for all $\mathcal{M}$, for all $((e^1,\ldots,e^\tau),x) \in \psi(\mathcal{M})$, where e^1 connects the components C_1 and C_2 into a component C, there exists an $\tilde{x} \in \mathbf{R}^C$ such that

$$((e^2,\ldots,e^\tau),(\tilde{x},x^{N\backslash C})) \in \psi(\mathcal{M}^{e^1}).$$

This property requires that if the first edge of a solution of an mcse problem is added to the initial graph, the solution to the reduced problem should include the remaining sequence together with an allocation which coincides with the original allocation for the players outside the component formed by adding this edge.

Max ψ satisfies *maximality* if for all $\mathcal{M}$, for all $(\mathcal{E},x) \in (E_{N^*})^\tau \times \mathbf{R}^N$ such that the solution ψ' defined by

$$\psi'(\mathcal{M}') = \begin{cases} \psi(\mathcal{M}) \cup \{(\mathcal{E},x)\} & \text{if } \mathcal{M}' = \mathcal{M} \\ \psi(\mathcal{M}') & \text{if } \mathcal{M}' \neq \mathcal{M} \end{cases} \tag{4.3.1}$$

satisfies Eff, MC, FSC, ET, ES and Loc, it holds that

$$(\mathcal{E},x) \in \psi(\mathcal{M}).$$

The upshot of this last property is that one should not be able to enlarge a solution without losing at least one of the previous properties.

Proposition 4.3.3 The proportional rule satisfies Eff, MC, FSC, ET, ES, Loc and Max.

Proof : The proportional rule satisfies Eff, FSC and MC because the set of allocations generated by the proportional algorithm is a refinement of the irreducible core and all allocations in the irreducible core satisfy these properties (cf. chapter 3). ET is a direct consequence of the definition of the proportional rule.

To prove that the proportional rule satisfies the equal share property, take an mcse problem $\mathcal{M}$ and take a component C of $\langle N^*, E\rangle$. Any two players i and $j \in C$ have the same initial obligations. For any sequence $\mathcal{E}$ constructed by the algorithm, the remaining obligations at a stage t are only dependent on the remaining obligations in the previous stage, so by an induction argument, i and j have the same remaining obligations throughout all stages. Since in the unique sequence of fractions $\mathcal{F}$ corresponding to $\mathcal{E}$ in the proportional algorithm, the fractions of edges that i and j pay are proportional to the remaining obligations, it follows that $f_i^t = f_j^t$ for all t and hence $x_i^{\mathcal{E},\mathcal{F}} = x_j^{\mathcal{E},\mathcal{F}}$. So the proportional rule has the equal share property.

The proportional rule is local : take $((e^1,\ldots,e^\tau),x) \in \mathrm{PRO}(\mathcal{M})$. Let e^1 connect two components C_1 and C_2 into a component C. Then $\tilde{\mathcal{E}} = (e^2,\ldots,e^\tau)$ leads to a minimum-cost spanning extension of $\mathcal{M}^{e^1}$. Let $\tilde{\mathcal{F}}$ be the unique sequence of fractions that corresponds to $\tilde{\mathcal{E}}$ in the algorithm 4.1.1 and define $\tilde{x} = x^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}}$. Then $\tilde{x}_k = x_k$ for $k \notin C$. Hence the proportional rule is local.

To prove the proportional rule satisfies Max, take an mcse problem $\mathcal{M}$ and take

$$((e^1, \dots, e^\tau), x) \in (E_{N^\bullet})^\tau \times \mathbf{R}^N$$

such that the solution PRO' as defined in equation 4.3.1 satisfies Eff, MC, FSC, ET, ES and Loc. Suppose e^1 connects the two components C_1 and C_2 into C. By locality, there exists an $\tilde{x} \in \mathbf{R}^C$ such that

$$((e^2, \dots, e^\tau), (\tilde{x}, x^{N \setminus C})) \in \mathrm{PRO}(\mathcal{M}^{e^1}).$$

Hence, there exist fractions vectors $(f^2, \dots, f^\tau)$ that are constructed by the proportional algorithm together with the sequence $(e^2, \dots, e^\tau)$, such that

$$(\tilde{x}, x^{N \setminus C}) = x^{(e^2, \dots, e^\tau), (f^2, \dots, f^\tau)}. \tag{4.3.2}$$

By efficiency of the proportional rule on $\mathcal{M}$ and $\mathcal{M}^{e^1}$,

$$\sum_{k \in C} (x_k - \tilde{x}_k) = w(e^1). \tag{4.3.3}$$

We now distinguish two cases :

1. If either of C_1 or C_2 (say C_1) contains the source, by FSC and equal treatment, we obtain
$$x_k - \tilde{x}_k = \begin{cases} 0 & \text{if } k \in C_1, \\ w(e^1)/|C_2| & \text{if } k \in C_2. \end{cases}$$
In this case, define f^1 by
$$f_k^1 = \begin{cases} 0 & \text{if } k \notin C_2, \\ 1/|C_2| & \text{if } k \in C_2. \end{cases}$$
Then $x = x^{(e^1, \dots, e^\tau), (f^1, \dots, f^\tau)}$ and $((e^1, \dots, e^\tau), x) \in \mathrm{PRO}(\mathcal{M})$.

2. If neither of C_1 or C_2 contain the source, by ET we obtain
$$x_k = \begin{cases} z_1 & \text{if } k \in C_1 \\ z_2 & \text{if } k \in C_2 \end{cases}$$
for some z_1 and z_2 satisfying
$$z_1|C_1| + z_2|C_2| = \sum_{k \in C} x_k = w(e^1) + \sum_{k \in C} \tilde{x}_k = w(e) + \sum_{k \in C} \sum_{t=2}^{\tau} f_k^t w(e^t),$$
by equations 4.3.3 and 4.3.2. Furthermore, by ES,
$$z_1|C_1| = z_2|C_2|.$$

Hence, $z_1 = \frac{w(e)+\sum_{k\in C}\sum_{t=2}^{\tau} f_k^t w(e^t)}{2|C_1|}$ and $z_2 = \frac{w(e)+\sum_{k\in C}\sum_{t=2}^{\tau} f_k^t w(e^t)}{2|C_2|}$, so defining $\tilde{\mathcal{F}}$ by

$$\tilde{f}_k^t = \begin{cases} f_k^t & \text{if } t>1 \text{ and } k \notin C, \\ 0 & \text{if } t=1 \text{ and } k \notin C, \\ \frac{\sum_{k\in C} f_k^t}{2|C_1|} & \text{if } t>1 \text{ and } k \in C_1, \\ \frac{\sum_{k\in C} f_k^t}{2|C_2|} & \text{if } t>1 \text{ and } k \in C_2, \\ \frac{1}{2|C_1|} & \text{if } t=1 \text{ and } k \in C_1, \\ \frac{1}{2|C_2|} & \text{if } t=1 \text{ and } k \in C_2, \end{cases}$$

we obtain $x = x^{(e^1,\ldots,e^\tau),\tilde{\mathcal{F}}}$. As $\tilde{\mathcal{F}}$ is the sequence of fraction vectors corresponding to $(e^1,\ldots,e^\tau)$ in the proportional algorithm applied to $\mathcal{M}$,

$$((e^1,\ldots,e^\tau),x) \in \mathrm{PRO}(\mathcal{M}).$$

This concludes the proof. □

Lemma 4.3.4 If a solution ϕ satisfies Eff, MC, FSC, ET, ES and Loc, and a solution ψ satisfies all these properties as well as Max, then $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ for all mcse problems $\mathcal{M}$.

Proof : Suppose not. Then there exists an mcse problem $\mathcal{M} = \langle N,*,w,E\rangle$ and $(\mathcal{E},x) \in \phi(\mathcal{M}) \setminus \psi(\mathcal{M})$. Including $(\mathcal{E},x)$ in $\psi(\mathcal{M})$ yields a solution that still has the properties Eff, MC, FSC, ET. Without loss of generality we assume that $\langle N^*,E\rangle$ has the least number of components of all problems with the property that $\phi(\mathcal{M}) \setminus \psi(\mathcal{M}) \neq \emptyset$. Then including $(\mathcal{E},x)$ in $\psi(\mathcal{M})$ does not violate ES nor Loc, so by maximality it follows that $(\mathcal{E},x) \in \psi(\mathcal{M})$. □

The next theorem implies the proportional rule is also the unique solution which satisfies all mentioned properties.

Theorem 4.3.5 The unique solution of mcse problems that satisfies Eff, MC, FSC, ET, ES, Loc and Max is the proportional rule.

Proof : We know that the proportional rule has the properties, and by lemma 4.3.4, if there are two solutions having them, they coincide. □

The decentralized rule has up to now not been characterized axiomatically, but it can be shown that it satisfies the properties Eff, MC, FSC, ET and ES. It does not satisfy Loc nor Max.

It remains an open problem to prove or disprove that the properties used to characterize the proportional rule are logically independent.

Chapter 5

Other network construction models

This chapter surveys four alternative models of the network construction problem. Our purpose her is not to give an exhaustive treatment of these models, but rather to make an open-ended presentation that will provide suggestions for further research.

Section 5.1 presents voluntary connection problems, in which the players want to be connected to the source only if this costs less than the value they assign to the service. Section 5.2 presents situations in which the source is unreliable; players will want to be connected to more sources, in order to be assured of the service. Section 5.3 discusses the case in which the cost of an edge between two nodes depends on the number of users that use this edge. Section 5.4 presents a noncooperative model of network construction.

5.1 Voluntary-connection games

The previous chapters used the assumption that the players want to get connected to the source at any cost. It might be, however, that an upper bound exists on the amount players are willing to pay for a connection with the source. This can be described by means of a model in which players can voluntarily connect to the source, in which case they obtain a reward.

This section, based upon a term paper by Marco van Bokhoven, presents such voluntary-connection problems. Consider an mcst situation $\langle N, *, w\rangle$, in which each player $i \in N$ has the extra possibility to use a local, private supplier (e.g. a well) instead of the central source. He prefers, however, to use the central source and has a positive benefit b_i from doing so. With this situation, associate a payoff game (N, u) in which the value of a coalition is the sum of the benefits of the players in this coalition, minus the cost of a minimum-cost tree connecting this coalition with the central source, in formula,

$$u(S) := \sum_{i \in S} b_i - c(S),$$

for all $S \subseteq N$, where (N, c) is the usual mcst game associated with $\langle N, *, w\rangle$. This game is a payoff game, of which the core is given by

$$\text{Core}(N,u) := \{x \in \mathbf{R}^N \mid \sum_{i\in N} x_i = u(N) \text{ and } x(S) \geq u(S) \text{ for all } S \subseteq N\}.$$

There is a bijection between the core of u and the core of c :

$$x \in \text{Core}(c) \Leftrightarrow b - x \in \text{Core}(u),$$

so the game u has a non-empty core.

However, for a coalition S it is sometimes better to connect only part of its members to the central supplier. Hence, the *voluntary connection game* (N, v) is defined by

$$v(S) := \max_{T\subseteq S} u(T).$$

It is a monotonic game : $v(T) \leq v(S)$ if $T \subseteq S$.

The game (N, u) is referred to as the *compulsory connection game*, to distinguish it from the voluntary-connection game. An obvious question is whether voluntary-connection games are balanced.

The following results are known.

Theorem 5.1.1 A voluntary-connection game associated with an mcst problem with at most three players is balanced and there is a marginal allocation in the core.

Proof : Here follows a sketch of the proof. Let $\langle N, *, w\rangle$ be an mcst situation with at most three players, and let $(b_i)_{i\in N}$ be the benefits derived by the players if connected to the source. Choose a coalition S with maximal worth $v(S)$ and a minimum-cost spanning tree $\langle S^*, E\rangle$ for S. Take an order σ of the players in N which preserves the order induced by the tree, i.e. if a player j is connected via player i to the source in the tree $\langle S^*, E\rangle$, then $\sigma_i < \sigma_j$. Furthermore, order the non-connected players last. Define the *marginal allocation* m^σ by

$$m_i^\sigma = v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma),$$

where $P_i^\sigma := \{j \mid \sigma(j) < \sigma(i)\}$ consists of the players who come before i in the order σ. Now the theorem follows from the following claim.

Claim 5.1.2 m^σ lies in the core of the game v.

For a proof of this claim, see Van Bokhoven (1994), who distinguishes several cases, depending on which tree the grand coalition N builds. □

For voluntary-connection games with four or five players it is unknown whether they are always balanced or not, but voluntary-connection games do exist with six players or more, which are not balanced.

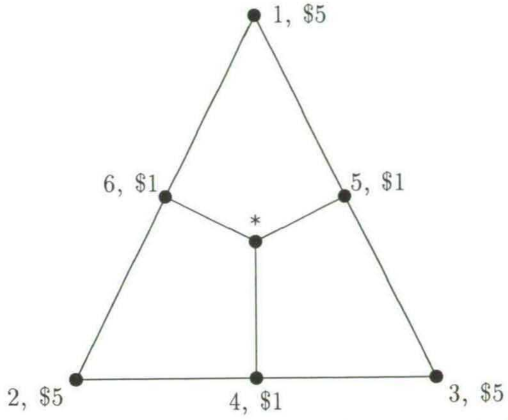


Figure 5.1: A voluntary-connection problem. The amounts in $ are the benefits.

Example 5.1.3 Consider the voluntary-connection game (N, v) associated with the mcst problem sketched in figure 5.1. The cost of the edges drawn is $2, the cost of edges not drawn is $100. Benefits are $b_1 = b_2 = b_3 = \$5$ and $b_4 = b_5 = b_6 = \$1$. It satisfies $v(N) = 7$ (connect all players except player 4), $v(126) = v(234) = v(135) = 5$ and $v(4) = v(5) = v(6) = 0$, hence the core is empty.

Note that in this example players can be added and placed at distance zero to player 1, to provide examples of non-balanced voluntary connection games with more than six players. Although the grand coalition does not connect itself fully in this example, it is not true that if the grand coalition connects itself, the voluntary-connection game is necessarily balanced. Jeroen Kuipers (private communication) constructed a seven-player voluntary-connection game with empty core, in which the grand coalition connects itself completely to the source.

5.2 Minimum-cost connecting forest problems

In a classical minimum-cost spanning tree problem, all users of a facility have to be connected to a unique source of the facility. In order to be supplied with a continuous service of the source, it is crucial that the source be reliable. Up to now, it was implicitly assumed that the source is perfect and not subject to failure.

This section, inspired by Van de Leensel (1994), drops this assumption and looks at the situation in which the source may fail. The reliability of the service can be improved by having more sources and building a network in which each player is connected to at least k sources, where k is an exogenously given natural number greater than or equal to 1. To such a situation, one can associate a cooperative cost game. We are interested in whether these games are balanced.

Let N be a set of users of a facility provided by sources in a set P, let $k \geq 1$ be an exogenously given natural number, and let $w : E_{N \cup P} \rightarrow \mathbb{R}_+$ be a weight function specifying for every edge e the positive cost of constructing this edge. The *minimum-cost connecting forest (mccf) problem* $\langle N, P, w, k\rangle$ is to construct a minimum-cost network that connects every player $i \in N$ with at least k sources in P and to allocate the cost of this network to the users of the facility. We will show that the optimal network is indeed a forest.

Denoting $|N| = n$ and $|P| = p$ it follows that a mccf problem makes sense only if $k \leq p$. Note that in the mcst problem $p = k = 1$.

5.2.1 Construction of an mccf

For an mccf problem $\langle N, P, w, k\rangle$, we call a vertex *k-source connected* in a graph $\langle N \cup P, F\rangle$ if it is connected to (at least) k sources.

First of all it is easy to see that an optimal graph for an mccf problem will never contain a cycle, so the problem deserves its name.

Lemma 5.2.1 An optimal graph for an mccf problem is a forest.

Proof : Suppose that we are given a graph in which all players are k-source connected and which contains a cycle. If we delete one of the edges in this cycle, we do not decrease the source connectivity of any of the vertices in the component of the solution containing the cycle. However, since we assumed that every edge has positive costs, we have reduced the total costs. Therefore, the original graph can not have been optimal. □

Kruskal's or Prim-Dijkstra's mcst algorithms can be applied if $|P| = 1$, which is the standard mcst problem. In order to generalize to the case of any given number of sources p (but $k = 1$), the algorithms need only minor modifications. We give an algorithm similar to Kruskal's algorithm.

Algorithm 5.2.2 (Van de Leensel, 1994)
input : an mccf problem $\langle N, P, w, 1\rangle$
output : an mccf $\langle N \cup P, F\rangle$

1. Set $F = \emptyset$.

2. Order the set $E_{N \cup P}$ of all edges in a list according to non-decreasing cost. Delete the edges connecting two sources. Take the first edge e from this list.

3. If adding e to F does not introduce a cycle, and e does not connect two components formed in $\langle N \cup P, F\rangle$ which both contain a source, then let $F := F \cup \{e\}$.

4. If not every vertex in N is 1-source connected in $\langle N \cup P, F\rangle$, take the next edge e on the list and go to step 3.

5. $\langle N \cup P, F \rangle$ is the required mccf.

In a similar way, one can generalize Prim-Dijkstra's algorithm and prove that these algorithms generate mccfs. A third way to construct an mccf for an mccf problem $\langle N, P, w, 1 \rangle$ is based on the following idea : add an extra vertex, $*$, and define the weights $w(\{*, p\}) = 0$ for $p \in P$, while $w(\{*, i\}) = M$, with M a large number ($M > w(e)$ for all $e \in E_{N \cup P}$). Now apply Kruskal's algorithm (algorithm 3.1.1) to the mcst problem $\langle N \cup P, *, w \rangle$. Delete $*$ and all edges of the form $\{*, p\}$ with $p \in P$ from the obtained mcst. This yields an mccf for the mccf problem.

Note that every player is connected to at most one source in an mccf for an mccf problem $\langle N, P, w, 1 \rangle$, because the cost of all edges is supposed to be positive.

So far we have considered problems requiring 1-source connectedness. One could also imagine a situation where each source has some unique capability or supplies a unique good. Then we might want to connect every vertex with every source. The following theorem states that the optimal solution for this case is a minimum-cost spanning tree.

Theorem 5.2.3 An mccf of an mccf problem $\langle N, P, w, |P| \rangle$ is a minimum-cost spanning tree of the mcst problem of the weighted graph $\langle N \cup P, E_{N \cup P}, w \rangle$.

Proof : Every vertex in N has to be connected to every source. Thus it is immediately clear that the mccf consists of one connected component. Lemma 5.2.1 tells us that the mccf is a forest. Now a connected forest is a spanning tree. □

For mccf problems which are not of the type $k = 1$ or $k = |P|$, a greedy algorithm obtaining mccfs like the ones above is as yet unknown. Van de Leensel (1994) does however give an integer program which constructs mccfs.

5.2.2 Allocation of the cost of an mccf

In this section, we will address the question of how to allocate the cost of an mccf to the players.

Inspired by Bird (1976), one can propose the following allocation rule ρ for mccf problems in which every player has to be connected to one source only. Given an mccf problem $\langle N, P, w, 1 \rangle$, take an mccf $\langle N \cup P, F \rangle$ and define $\rho^F \in \mathbb{R}^N$ as the allocation which assigns to every player i the cost of the edge incident with i on the unique path in the forest $\langle N \cup P, F \rangle$ from i to a source.

This allocation is a core element of the associated mccf game, which we now define.

Definition 5.2.4 For an mccf problem $\langle N, P, w, k \rangle$, the associated mccf game (N, c) is defined by setting the cost of a coalition $S \subseteq N$ equal to the cost of an mccf of the mccf problem $\langle S, P, w, k \rangle$.

Theorem 5.2.5 For an mccf $\langle N \cup P, F\rangle$ of the mccf problem $\langle N, P, w, 1\rangle$, ρ^F is in the core of the mccf game associated with this mccf problem.

Proof : That ρ^F is efficient is trivial since each edge in the forest is allocated to a player. It remains to show that no coalition can protest. Consider the optimal solution for an arbitrary coalition S. This is a forest in which each component contains one source and a number of players (possibly zero) in S. Add the players (vertices) of $N \setminus S$ to this graph as well as the edges that are allocated to the players in $N \setminus S$ by ρ^F. Since the edges allocated to $N \setminus S$ do not form a cycle, there is an edge from one of the players in $N \setminus S$ that is connected with either a player in S or a source. Thus we already have found that after adding this edge to the optimal solution for S, all players in S and one player in $N \setminus S$ are 1-source connected. Repeating this argument for the remaining players in $N \setminus S$ implies that the resulting graph is a forest in which each player of N is 1-source connected. Since the forest created in this way does not have to be optimal for N, we have that

$$c(N) \leq c(S) + \sum_{i \in N\setminus S} \rho_i^F = c(S) + \rho^F(N \setminus S).$$

Efficiency of ρ gives us that $c(N) = \rho^F(S) + \rho^F(N \setminus S)$. Combining this with the above inequality implies $\rho^F(S) \leq c(S)$. □

For an mccf problem $\langle N, P, w, |P|\rangle$ in which every player has to be connected to every source, Van de Leensel provides core elements $(\lambda_{pg})_{p \in P, g \in N}$ of the associated mccf game in the following way. Choose a player $g \in N$, who will be responsible for edges of which it is not immediately clear who should pay them. Choose an arbitrary source p. Note that in this mccf problem, an mccf is an mcst. Construct an mcst following Prim-Dijkstra's algorithm, starting at source p. At every step, if the edge added connects a *player* to the component of source p, this player is held responsible for the cost of this edge; if instead the edge added connects another *source* to the component of p, then g has to pay the cost of this edge. It will be clear that this allocation λ_{pg} is very discriminating towards player g. Let us now show that the suggested allocations are located in the core of the mccf game.

Theorem 5.2.6 For every player $g \in N$, for every source $p \in P$, λ_{pg} is a core element of the mccf game (N, c) associated with the mccf problem $\langle N, P, w, |P|\rangle$.

Proof : It is immediately clear that λ_{pg} is efficient, since every edge that is added during the algorithm is immediately allocated to some player. Consider the mccf for the mccf problem of an arbitrary coalition S. This is a tree containing all the players in S and all the sources. Add the players (vertices) of $N \setminus S$ to this graph as well as the set H of those edges which connect a player in $N \setminus S$ to the component of p when constructed by Prim-Dijkstra's algorithm. (Do not add the edges which are allocated to player g because they connect a source with the component of p). Since the edges in H do not form a cycle, there is an edge in H that is incident with either a player of S or a source.

Hence, after adding this edge to the mcst for S, all players in S and one player in $N \setminus S$ are $|P|$-source connected. Repeating this argument for the remaining players in $N \setminus S$ (and the remaining edges in H) implies the graph is a spanning tree, in which each player is $|P|$-source connected.

Since the tree created in this way does not have to be optimal for N, we have that

$$c(N) \leq c(S) + \sum_{e \in H} w(e).$$

Now we have to consider two cases. If $g \in N \setminus S$ then $\sum_{e \in H} w(e) < \lambda(N \setminus S)$. If on the other hand $g \in S$ then $\sum_{e \in H} w(e) = \lambda(N \setminus S)$. Hence we can conclude that $\sum_{e \in H} w(e) \leq \lambda(N \setminus S)$. Together with the efficiency of λ this then gives us

$$\lambda(S) + \lambda(N \setminus S) \leq c(s) + \lambda(N \setminus S),$$

which implies $\lambda(S) \leq c(S)$. □

It follows that mccf games associated to mccf problems in which all players have to be connected to all sources are balanced.

In the preceding it has become clear that for two cases the core of a mccf game associated with an mccf problem $\langle N, P, w, k \rangle$ is non empty:

- $|P| \geq 1$, $k = 1$,
- $|P| = k$.

A third case in which the core is non-empty is the case

- $|P| \geq k$, $|N| \leq 2$,

because an mccf game (N, c) is a *subadditive game*, i.e. $c(S) + c(T) \leq c(S \cup T)$ for disjoint coalitions S and T. For the remaining cases, it turns out that the core of the associated games may be empty.

The smallest problem not belonging to one of the types above has three sources, three users and every user has to be connected with at least 2 sources.

Example 5.2.7 Consider the mccf problem with three players, three sources and in which every player has to be connected with two sources, of which the graph is sketched in figure 5.2. Sources are indicated by squares, players by disks. It follows that $c(\{i, j\}) = 12$ for every pair $\{i, j\}$ and $c(N) = 23$. This game is not balanced. Suppose there was a core element x. Then

$$23 = c(N) = x_1 + x_2 + x_3 = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 + x_3) + \frac{1}{2}(x_2 + x_3) \leq 18,$$

which is a contradiction. Therefore the core is empty.

This example can be modified to show that for more players and/or more sources the core can also be empty if $1 < k < |P|$.

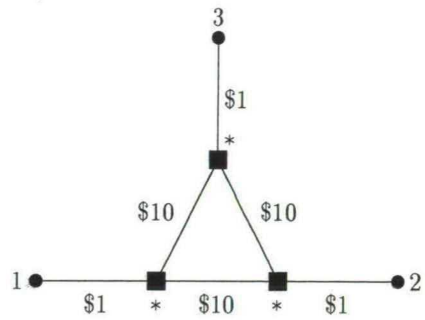


Figure 5.2: Edges not indicated are expensive.

5.3 Connections with variable costs

Up to now, we assumed that the cost of a connection is independent of the number of users of this conection. In this section, which is based upon a term paper of Van der Leeden (1994) we relax this assumption and assume the cost of constructing an edge varies with the number of users of the edge.

This yields the following situation. A set N of agents wants to be connected to a facility, of which the source is denoted $*$. There is a vector valued weight function $w : E_{N^*} \to \mathbb{R}^{|N|}$, which determines for an edge e the cost vector $w(e) = (w_k(e))_{k=1}^{|N|}$. Here, $w_k(e)$ is interpreted as the cost of constructing a link along edge e which can provide service to k users. We call such a situation a *minimum cost connecting graph problem.*

A solution to the minimum-cost connecting graph (*mccg*) problem has to specify which edges are constructed, as well as for each edge, which agents use this edge, in order to be able to tell what the cost of the graph is.

Without further restrictions a minimal cost connecting graph for an mccg problem can have cycles, see example 5.3.1.

Example 5.3.1 Consider the mccg problem, with set of agents $N = \{1, 2, 3\}$, of which the costs of edges are given in table 5.1. A solution for this problem is given in figure 5.3.

e	{*,1}	{*,2}	{*,3}	{1,2}	{1,3}	{2,3}
$w(e)$	(4,5,6)	(2.5,5,6)	(1,1,3)	(1,2,3)	(4,5,7)	(2,3,5)

Table 5.1: An mccg problem

The numbers next to an edge indicate how many players use the edge. In fact, two solutions are compatible with this figure. In the first solution, player 3 is directly connected to the source, player 2 is connected via player 3, and player 1 is connected via player 2. In the second solution, both players 2 and 3 are directly connected to the source, and player 1 is connected via players 2 and 3.

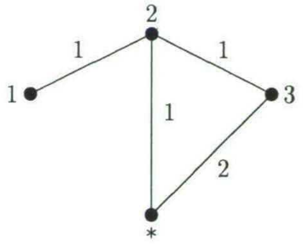


Figure 5.3: An mccg with a cycle.

With an mccg problem $\langle N,*,w\rangle$, we associate an mccg game (N,c), defined by assigning to a coalition S the cost $c(S)$ of an mccg of the problem $\langle S,*,w\rangle$.

In the special case where the cost of an edge is independent of the number of users, the mccg problem becomes an mcst problem, and Bird (1976) proved that the core of the mccg game is not empty. In the general case, the core of an mccg game can be empty.

e	{*,1}	{*,2}	{*,3}	{1,2}	{1,3}	{2,3}
$w(e)$	(10,11,100)	(10,11,100)	(10,11,100)	(1,2,3)	(1,2,3)	(1,2,3)

Table 5.2: Costs of edges in an mccg problem.

Example 5.3.2 Consider the mccg problem, with set of agents $N = \{1,2,3\}$, of which the costs of edges are given in table 5.2. The mccg game is given in table 5.3 and has an empty core.

Hence we are interested in properties of the weight function which guarantee the mccg games to be balanced. One property that guarantees balancedness is linearity of the weight function. Here, a weight function w is said to be linear if $w_k(e) = kw_1(e)$ for all edges e and all $k \leq |N|$.

Theorem 5.3.3 An mccg game (N,c) associated with an mccg problem $\langle N,*,w\rangle$ with linear weights is balanced.

Proof : For every player $i \in N$, find the minimum-cost graph $\langle N^*, E_i\rangle$ connecting this player to the source, possibly via other players. Here, the cost of a graph $\langle N^*, E\rangle$ equals the sum $\sum_{e\in E} w_1(e)$, because only player i uses the connections. Define $\underline{c}_i$ to be the cost of this minimum-cost graph.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$c(S)$	10	10	10	12	12	12	22

Table 5.3: An mccg game with empty core.

Define the additive game $(N, \underline{c})$, by putting for every coalition S

$$\underline{c}(S) := \sum_{i \in S} \underline{c}_i = \sum_{i \in S} \sum_{e \in E_i} w_1(e).$$

Note that the graph $\langle N^*, \bigcup_{i \in S} E_i \rangle$ is a graph that connects every player in S to the source, but that maybe vertices outside S are used.

Now a solution to the mccg problem for S specifies for every edge who uses it. This implies it specifies for every user $i \in S$ a graph $\langle S^*, F_i^S \rangle$ representing the path connecting i via players in S to the source. Hence,

$$c(S) = \sum_{i \in S} \sum_{e \in F_i^S} w_1(e)$$

because the weights of links are linear. Because the graph $\langle N^*, E_i \rangle$ was a minimal cost graph connecting i to the source, it follows that $c(S) \geq \underline{c}(S)$ for every coalition S. Finally, because the path F_i^N can use all players in N, it follows that $\sum_{e \in F_i^N} w_1(e) = \sum_{e \in E_i} w_1(e)$, which implies $c(N) = \underline{c}(N)$. Hence, the core of $\underline{c}$ is included in the core of c. Now the allocation x, defined by $x_i = \underline{c}_i$ for every player $i \in N$, is a core element of $\underline{c}$ and hence it is a core element of the mccg game. □

It is still an open question to find more general conditions which guarantee the mccg game to be balanced.

5.4 A non-cooperative approach

Except in section 2.3, all network construction games presented up to now were cooperative games. We here present another non-cooperative network construction model, suggested by Jose Zarzuelo.

With an mcst situation $\langle N, *, w \rangle$, associate a strategic *path game* $\langle N, (A^i)_{i \in N}, (u_i)_{i \in N} \rangle$ in which an action a^i of a player i is to choose a graph $\langle N^*, E_i \rangle$ connecting i (via other players) to the source, and given the actions $a = (a^j)_{j \in N}$ of the players, the payoff to player i equals

$$u_i(a) := -\sum_{e \in E_i} \frac{w(e)}{|\{j \in N \mid e \in E_j\}|}.$$

The name path game is motivated by the fact that in a Nash equilibrium of a path game, a player will choose a graph that contains only edges that lie on a specific path from i to the source, and edges with cost zero.

One can show that these path games have pure Nash equilibria, by showing that they are congestion games.

Definition 5.4.1 (Rosenthal, 1973)) A *congestion situation* $\langle N, M, (O_i)_{i \in N}, c \rangle$, consists of a set N of users and a set of primary factors M, which the users can use to attain their objective. The objective of a player i is represented by the set $O_i \subseteq 2^M$ of collections of primary factors with which i's objective can be attained. The function $c : \mathbb{N} \times M \to \mathbb{R}_+$ specifies that the cost of k users using a primary factor m is $c(k, m)$.

The *congestion game* $\langle N, (X_i)_{i\in N}, (u'_i)_{i\in N}\rangle$ associated with a congestion situation, is defined as follows. An action of a player i is a collection $M_i \in O_i$ of primary factors that attains i's objective, while the payoff to player i at strategy profile $x = (M_j)_{j\in N}$ equals

$$u'_i(x) := -\sum_{m\in M_i} c(|\{j \in N \mid m \in M_j\}|, m).$$

Rosenthal (1973) proved that congestion games have a pure Nash equilibrium. In fact, they are potential games (see Monderer and Shapley, 1993).

For an mcst situation $\langle N, *, w\rangle$, we define an associated congestion situation $\langle N, M, (O_i)_{i\in N}, c\rangle$, by

- $M = E_{N^*}$, the set of all edges in the complete graph on N.
- For each player i, the set O_i equals the set of all graphs in which i is connected to the source.
- $c(k, e) = \frac{w(e)}{k}$.

It is clear that the congestion game associated with this congestion situation coincides with the path game. Of course, in a congestion game, one would expect the cost of a primary factor to go up if it is used by more players, but this is not essential.

Corollary 5.4.2 A path game has pure Nash equilibria.

It is not true, though, that these Nash equilibria always correspond to minimum-cost spanning trees.

Example 5.4.3 Reconsider the mcst problem of example 1.0.1, drawn in figure 1.1. The minimum cost spanning tree consists of the edges {Spring, Ann}, {Ann, Charley} and {Charley, Bart}. Now this graph is not a Nash equilibrium, because if this is the resulting graph, then Ann has chosen the edge {Spring, Ann}, Charley has chosen the edges {Spring, Ann} and {Ann, Charley}, and Bart has chosen the edges {Spring, Ann}, {Ann, Charley} and {Charley, Bart}. This implies Bart has a payoff of $-40/3-26/2-35$, which is less than the -50 he would get if he deviated and connected himself directly to the source.

Other types of strategic network construction games can be associated with the voluntary-connection games of section 5.1, or the games associated to network construction games with variable costs of edges of section 5.3.

Part II

Veto Control and Cooperation

Part II treats economic situations in which revenue can be generated using resources that are controlled by agents, and asks how the revenue accrued will be allocated to the agents. These situations are analyzed by means of cooperative payoff games. Chapter 6 presents a linear production economy with transportation possibilities, in which the primary goods are the controlled resources. Chapter 7 presents a situation in which a network can generate revenue and the vertices and edges of this network are the controlled resources. Chapter 8 presents controlled economic situations that generalize the two previous models.

The control involved is modeled by means of control games, introduced by Curiel, Derks, and Tijs (1989). A *control game* is a TU-game in which every coalition has a worth of either zero or one, and the grand coalition has worth one. These games generalize monotonic simple games, defined by von Neuman and Morgenstern (1944) and used by Shapley (1962). Dubey (1975) defines a *simple game* (N, v) as a TU-game in which the range of the characteristic function v is $\{0, 1\}$. A *monotonic simple game* is a simple game (N, v) such that $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. Note that a non-zero monotonic simple game is automatically a control game and that a simple game v is completely determined by the set

$$W(v) := \{S \subseteq N \mid v(S) = 1\}$$

of *winning coalitions.*

This study of control games leads to an axiomatic characterization of the Shapley value and the Banzhaf value on the class of control games, on the class of simple games and on the class of all TU-games. This is presented in chapter 9.

Finally, chapter 10 presents a simple and efficient algorithm that computes the nucleolus and the kernel of games with veto players. A veto player of a non-negative TU-game is a player whose absence prevents a coalition from obtaining a non-zero payoff. Veto players appear in various economic situations, such as markets with a monopolist.

We will now include a few standard notations and definitions. The class of TU-games with player set N will be denoted by G^N, the class of simple games with player set N by SG^N, the class of control games with player set N by CG^N and the class of monotonic simple games with player set N by MSG^N.

Examples of control games are the unanimity games. For each $S \in 2^N \setminus \{\emptyset\}$, the *unanimity game* (N, u_S) is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{if } S \not\subseteq T, \end{cases}$$

for all $T \subseteq N$.

It is well known that the collection $\{(N, u_S) \mid S \in 2^N \setminus \{\emptyset\}\}$ of all unanimity games forms a basis of the vector space G^N of all TU-games.

A game is *non-negative* if the worth of each coalition is non-negative. It is a *positive game* if the worth of each non-empty coalition is positive.

A player $i \in N$ is called a *veto player* of a TU-game (N, v), denoted by $i \in \text{veto}(v)$, if i can block a non-zero outcome, i.e. $v(S) = 0$ for all coalitions S not containing player i. A TU-game (N, v) is a *veto-rich game* if it has at least one veto player i.

A TU-game is *N-monotonic* if $v(N) \geq v(S)$ for all S.

A *superadditive game* (N, v) satisfies $v(S) + v(T) \geq v(S \cup T)$ for any two disjoint coalitions S and T. A *subadditive game* (N, v) satisfies $v(S) + v(T) \leq v(S \cup T)$ for any two disjoint coalitions S and T. An *additive game* is a game that is both superadditive and subadditive.

Often, it is assumed the grand coalition N of a TU-game forms and its worth has to be divided among the players. If each individual player i is rational, i will refuses allocations which assign less to i than i can obtain alone, the remaining allocations form the *set of imputations* $I(N, v)$ of a TU-game (N, v), defined as

$$I(N, v) := \{x \in \mathbf{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

Note that the imputation set of an arbitrary game can be empty. An *inessential game* (N, v) is a game with exactly one imputation, i.e. $v(N) = \sum_{i \in N} v(\{i\})$, any other game is called *essential*. An example of an inessential game is a *zero game* (N, v), defined by $v(S) = 0$ for all coalitions S.

If in addition we assume *coalitional rationality*, i.e. a coalition S will reject an allocation in which S does worse than S could do by seceding, one obtains the *core* of the game, which is for a payoff game (N, v) translates into

$$\text{Core}(N, v) := \{x \in I(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

A theorem by Bondareva (1963) and independently by Shapley (1967) states that a game has a non-empty core if and only if it is *balanced*. As in part II we never explicitly use the condition of balancedness, we do not define it. We do, however, call a game with a non-empty core balanced.

If not only the game (N, v) but also all its subgames (T, v^T) $(T \subseteq N)$ are balanced, then the game (N, v) is said to be a *totally balanced game*. Here, a *subgame* (T, v^T) is defined by $v^T(S) = v(S)$ for all $S \subseteq T$.

Sprumont (1990) argued that providing a core element is not enough and that a solution should also determine what happens if a subcoalition forms instead of the grand coalition. For this reason, he introduced a *population monotonic allocation scheme (PMAS)* of a game. A PMAS of a game (N, v) is a collection $x = \{x_{jS} \mid j \in S \subseteq N\}$ which satisfies the following two conditions

- $x_S(S) := \sum_{j \in S} x_{jS} = v(S)$ for all $S \subseteq N$.

- $x_{jS} \leq x_{jT}$ if $j \in S \subseteq T$.

Sprumont (1990) proves that a TU-game (N, v) which has a PMAS x is totally balanced. For example, $(x_{iS})_{i \in S}$ is a core element of the subgame (S, v_S) for every coalition S.

The *Shapley value* ϕ (cf. Shapley (1953)) of a game $v \in G^N$ is defined by

$$\phi_i(v) = \sum_{S: i \in S} \frac{|N \setminus S|!\,|S \setminus \{i\}|!}{|N|!}[v(S) - v(S \setminus \{i\})]$$

for all $i \in N$. As before, $|S|$ denotes the cardinality of the set S. The Shapley value is a linear map that satisfies

$$\phi_i(u_S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

for all $i \in N$ and all $S \subseteq N$.

A *directed graph* or *digraph* D on a set of vertices V is a subset of $V \times V$. An *arc* is an ordered pair $(v, w) \in V \times V$. For a vertex $v \in V$, we denote $D(v) := \{w \in V \mid (v, w) \in D\}$ and $D^{-1}(v) := \{u \in V \mid (u, v) \in D\}$. A digraph D on V is *reflexive* if $(v, v) \in D$ for all $v \in V$. A digraph D on V is *transitive* if $(u, v) \in D$ and $(v, w) \in D$ imply $(u, w) \in D$.

For two vectors $x, y \in \mathbf{R}^A$, with A an arbitrary finite set, we denote by $\langle x, y \rangle$ the inner product $\sum_{a \in A} x_a y_a$ of the two vectors.

Chapter 6

Controlled linear production with transportation possibilities

Owen (1975b), Granot (1986) and Curiel, Derks, and Tijs (1989) analyzed linear production (LP) games. These are transferable utility games associated with the following type of situation. There is one facility at which a linear production technology is available. A finite number of agents control the resources needed for production. Prices of the products are fixed exogenously. In the corresponding LP-game, the worth of a coalition of agents is the maximal value of a bundle it can produce with the resources it controls. Sufficient conditions were given for the LP-game to be balanced, i.e. to have a non-empty core. Moreover, it was shown that if these conditions are satisfied, there is a core element of the LP-game which can be computed by solving only the dual of the linear program which determines the worth of the grand coalition.

Koster (1990) analyzed a similar type of situation involving two facilities and transport of products, resources and technology from one facility to the other. The same conditions as in the one-facility case ensure the corresponding LP-game to be balanced.

This chapter, which is based on Feltkamp, Van den Nouweland, Borm, Tijs and Koster (1993), generalizes Koster's model. We consider situations with a finite number of facilities, each with its own linear production technology and exogenously fixed prices on products. We assume the markets are insatiable : every product manufactured or imported at a facility can be sold at its price at that facility. Furthermore, there are no capacity restraints, but at each facility only a finite amount of resources is available, which is controlled by the players. The facilities are public goods : use of a facility by a coalition does not inhibit its use by another coalition. A linear cost is associated to the use of the technology of a facility.

If these production sites were isolated, nothing new would be obtained. However, we allow transport of products, resources and technology between the facilities, along exogenously given routes. The possible transport routes for products, resources and technologies are represented by arcs of directed graphs. Successive transportation via two arcs is not allowed. This is not a loss of generality : it is possible to take transitive transport graphs, i.e. transport graphs such that if transport from f to g and from g to h is possible then also transport from f to h is possible.

Finally, we assume there are linear transport costs and linear losses of goods during transport. In the corresponding LPT-game, each coalition of players tries to produce a bundle of maximal worth with the resources it controls, possibly transporting resources, products and technologies to take advantage of opportunities at every site.

This chapter is organised as follows. Section 6.1 presents Owen's (1975b) model. In section 6.2 an example of a linear production situation with transportation possibilities is presented. In section 6.3 LPT situations are formally presented. It is shown that under certain conditions on the control over resources, the associated LPT-game is balanced, and that a core element can be found by solving only the dual of the linear program of the grand coalition.

6.1 Linear production situations

Owen (1975b) introduced linear production situations (LP situations for concision). A linear production situation consists of a set N of players, who can exploit a facility to produce a collection P of products using a collection R of resources. Production is linear, i.e. there are numbers $(a_{rp})_{p\in P, r\in R} \geq 0$ specifying that a_{rp} units of resource r are needed for production of one unit of good p. The numbers $(a_{rp})_{p\in P, r\in R}$ form a *technology matrix* A. It is assumed that in every column of A there is at least one positive entry, which is interpreted as : no product can be produced without using at least one resource. Each player $i \in N$ owns a bundle $b_i \in \mathbb{R}^R_+$. There is a price vector $c \in \mathbb{R}^P_+$ specifying the prices of products.

With such an LP situation, Owen associated a linear production game (N, v), in which the worth of a coalition S equals the maximal value of a production plan using the pooled resources of the players in S, in formula,

$$v(S) := \max\{\sum_{p\in P} x_p c_p \mid Ax \leq \sum_{i\in S} b_i\}.$$

The following holds.

Theorem 6.1.1 (Owen, 1975a) A linear production game is balanced and a core element can be generated by solving only one linear program, viz. the dual program of the linear program which gives the worth of the grand coalition.

Granot (1986) and Curiel, Derks, and Tijs (1989) generalize the linear production situations by allowing a more general control of the players over the resources. We will not go into detail here.

6.2 An example

In this section, we present an example of a linear production situation with transport. Consider three facilities, f, g and h, which together allow linear production of five products $p_1, \ldots, p_5$, using two resources r_1 and r_2.

At facility f only products p_1 and p_2 can be manufactured. Producing one unit of p_1 requires one unit of r_1 and three units of r_2, while manufacturing one unit of p_2 requires no units of r_1 and two units of r_2. We represent these technology constraints by a *technology matrix* A^f of which the first column corresponds to product p_1, the second column to p_2 and the rows correspond in a similar way to the resources r_1 and r_2, respectively. So,

$$A^f = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix},$$

and production of a bundle $q = (\ q_1 \quad q_2\ )^\top$ of the products p_1 and p_2 at facility f requires the resource bundle $A^f q = (\ q_1 \quad 3q_1 + 2q_2\ )^\top$. Here, with $M^\top$ we denote the transpose of a matrix M.

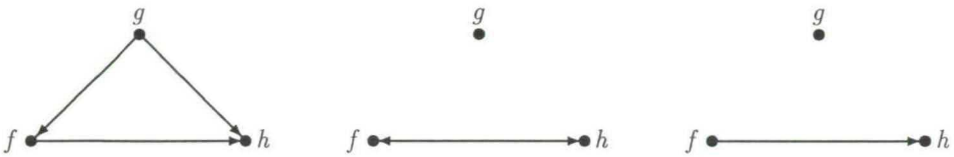
At facility g products p_3 and p_4 can be manufactured, so the technology matrix A^g has columns corresponding to p_3 and p_4 and rows corresponding to r_1 and r_2. At facility h product p_5 can be manufactured, so the column of A^h corresponds to p_5 and the rows correspond to r_1 and r_2. The technology coefficients in these matrices are as follows.

$$A^g = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}, \qquad A^h = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

At each facility there is an exogenously given vector of prices at which products can be sold at that facility. The price vector at facility f is $c^f = (4, 1, 4, 1, 4)$, which means that at f a bundle $q = (\ q_1 \ \ \ldots \ \ q_5\ )^\top$ of products has worth $4q_1 + q_2 + 4q_3 + q_4 + 4q_5$, which we denote by $\langle c^f, q\rangle$. Implicitly we assume the markets are insatiable : everything produced can be sold. Similarly, the price vectors at g and h are $c^g = (1, 3, 1, 3, 1)$ and $c^h = (2, 1, 2, 1, 2)$ respectively. Note that a price is specified for each product at each facility. The structure of the price vectors and the similarity of the production matrices at f and g can be thought of as due to products p_1, p_3 and p_5 being close substitutes produced at different facilities. A similar argument explains the equality in prices for products p_2 and p_4. Slightly abusing notation, if $P' \subseteq P = \{p_1, \ldots, p_5\}$, $q \in \mathbf{R}^{P'}$ and $c \in \mathbf{R}^P$ is a price vector, we will write $\langle c, q\rangle$ instead of $\sum_{p \in P'} c_p q_p$.

There are two players, called 1 and 2, each of whom owns a bundle of resources at each facility. Player 1 owns the bundles $b^f(1) = (0, 5)^\top$, $b^g(1) = (3, 0)^\top$ and $b^h(1) = (2, 0)^\top$ at f, g and h respectively, while player 2 owns the bundles $b^f(2) = (0, 3)^\top$, $b^g(2) = (1, 2)^\top$ and $b^h(2) = (0, 3)^\top$ at f, g and h respectively. The players can cooperate by pooling their resources.

Players can transport products, resources and technologies according to the following rules. Transport costs are zero. Resource transport is possible from g to f and h, and from f to h. Product transport is possible from f to h and vice versa and technology transport is possible from f to h. These transport possibilities are modeled by means of transport graphs (see figure 6.1).



Resource transport Product transport Technology transport

Figure 6.1: The transport graphs

We will first analyze the situation in which no transport is possible, and then gradually include transport possibilities. With such a LP situation a cooperative transferable utility (TU) game (N, v) is associated in the following way. The player set is $N = \{1, 2\}$, and a coalition $S \subseteq N$ has value $v(S)$ equal to the maximal revenue it can obtain through the sale of goods produced with resources the members of S own.

First, suppose there is no transport at all. In order to know what player 1 can obtain from production at facility f, we have to solve the linear program

$$\max\{\langle c^f, q^f\rangle \mid A^f q^f \leq b^f(1),\ q^f \geq 0,\ q^f \in \mathbf{R}^{\{p_1,p_2\}}\}.$$

The value of this program is 2.5. Similarly, at g she can obtain

$$\max\{\langle c^g, q^g\rangle \mid A^g q^g \leq b^g(1),\ q^g \geq 0,\ q^g \in \mathbf{R}^{\{p_3,p_4\}}\} = 0,$$

and at h,

$$\max\{\langle c^h, q^h\rangle \mid A^h q^h \leq b^h(1),\ q^h \geq 0,\ q^f \in \mathbf{R}^{\{p_5\}}\} = 0.$$

As there is no interaction between the three facilities, we can total these three revenues to obtain $v(\{1\}) = 2.5$. Similarly, we compute $v(\{2\}) = 4.5$, and $v(\{1, 2\}) = 9$. Because the feasible regions of the linear programs vary from coalition to coalition and the prices are constant, it may be easier to compute the value of the dual programs, which have the same feasible region for all coalitions.

If players can transport *technology* along the routes depicted in figure 6.1, they can manufacture products at h using either the production techniques represented in the technology matrix A^f or the techniques represented in technology matrix A^h. Accordingly, we replace the technology matrix A^h by the matrix $\bar{A}^h = (A^f, A^h)$. The other technology matrices are unchanged : $\bar{A}^f = A^f$, $\bar{A}^g = A^g$. We are again in the same sort of situation as when there was no transport at all, except that now

$$q^h \in \mathbf{R}^{\{p_1,p_2,p_5\}} \qquad \text{and} \qquad \bar{A}^h = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

Denoting the corresponding characteristic function by v_T, we see that

$$v_T(\{1\}) = 2.5, \qquad v_T(\{2\}) = 6, \qquad v_T(\{1,2\}) = 10.$$

Now, suppose players can also transport *products* along the routes depicted in figure 6.1. Then a good produced at facility h can be sold at either facility f or facility h, hence it can be sold for the maximum of the price at f and the price at h. The same goes for products manufactured at f, hence we replace the price vectors c^f and c^h with the (coordinatewise) maximum of c^f and c^h. Denote these new price vectors by $\bar{c}^h = \bar{c}^f$. The only change from the previous situation is that now $\bar{c}^h = (4,1,4,1,4)$. Denote the corresponding characteristic function by v_{TP}. Computing the worths of the coalitions yields

$$v_{TP}(\{1\}) = 2.5, \qquad v_{TP}(\{2\}) = 6, \qquad v_{TP}(\{1,2\}) = 12.6.$$

Finally, suppose transport of resources is also possible. In contrast to the previous cases, this program cannot be solved by three separate linear programs. We assume that successive transportation along two arcs is not possible. For example, from facility f only resources present at f can be transported to h. Denoting $t^{f^1 f^2}$ the bundle of resources transported from a facility f^1 to a facility f^2, one can see that the optimization problem player 1 now faces is

$$\max \langle \bar{c}^f, q^f \rangle + \langle \bar{c}^g, q^g \rangle + \langle \bar{c}^h, q^h \rangle$$

$$\begin{array}{lrcl}
\text{s. t.} & \bar{A}^f q^f & \leq & b^f(1) + t^{gf} - t^{fh}, \\
& \bar{A}^g q^g & \leq & b^g(1) - t^{gf} - t^{gh}, \\
& \bar{A}^h q^h & \leq & b^h(1) + t^{fh} + t^{gh}, \\
& q^f, q^g, q^h & \geq & 0, \\
& t^{fh}, t^{gf}, t^{gh} & \geq & 0, \\
& t^{gf} + t^{gh} & \leq & b^g(1), \\
& t^{fh} & \leq & b^f(1),
\end{array}$$

which has the value 12. Hence, denoting the characteristic function corresponding to this situation by v_{TPR}, we see $v_{TPR}(1) = 12$. The last equation is due to the condition that no transportation of resources from g via f to h is possible. Player 2 faces a similar linear program with value $v_{TPR}(2) = 8.5$ and coalition $\{1,2\}$ has to solve

$$\max \langle \bar{c}^f, q^f \rangle + \langle \bar{c}^g, q^g \rangle + \langle \bar{c}^h, q^h \rangle$$

$$\begin{array}{lrcl}
\text{s. t.} & \bar{A}^f q^f & \leq & b^f(1) + b^f(2) + t^{gf} - t^{fh}, \\
& \bar{A}^g q^g & \leq & b^g(1) + b^g(2) - t^{gf} - t^{gh}, \\
& \bar{A}^h q^h & \leq & b^h(1) + b^h(2) + t^{fh} + t^{gh}, \\
& q^f, q^g, q^h & \geq & 0, \\
& t^{fh}, t^{gf}, t^{gh} & \geq & 0, \\
& t^{gf} + t^{gh} & \leq & b^g(1) + b^g(2), \\
& t^{fh} & \leq & b^f(1) + b^f(2),
\end{array}$$

which yields $v_{TPR}(\{1,2\}) = 21.4$.

6.3 LPT-games

Consider a finite set $N = \{1, \ldots, n\}$ of players, who can make use of a finite set F of facilities to manufacture products. The (finite) set of resources is denoted by R, the (finite) set of all products by P and the subset of those products which a facility $f \in F$ can manufacture by P^f. For convenience, assume the sets P^f are disjoint. This harmless assumption can be satisfied by adding an index to every product, which varies according to the facility where the product is produced. Production is linear, i.e. there exist nonnegative numbers $(a^f_{rp})_{r\in R, p\in P^f}$ for each facility f, such that production of q_p units of product $p \in P^f$ at facility f, requires $a^f_{rp}q_p$ units of resource r as input. Hence, production of a bundle $q \in \mathbb{R}^{P^f}_+$ of products at facility f requires the bundle of resources $A^f q$, where $A^f = (a^f_{rp})_{r\in R, p\in P^f}$ is the technology matrix at f. Assume that for each facility f, and for each product $p \in P^f$, there is at least one resource r with $a^f_{rp} > 0$. This means that no product can be created out of nothing.

At every facility f, there is an exogenously given price vector $c^f \in \mathbb{R}^P$. We assume the markets in which the products are sold to be insatiable, i.e. every product p produced at (or transported to) a facility f can be sold at f for the price c^f_p. Hence, with some abuse of notation, a bundle $q \in \mathbb{R}^{P^f}$ is worth

$$\langle c^f, q\rangle = \sum_{p\in P^f} c^f_p q_p$$

at facility f.

The resources available are controlled by the players in the following way. For each coalition $S \in 2^N \setminus \{\emptyset\}$ and each facility f, there is a bundle $b^f(S) \in \mathbb{R}^R_+$ of resources S can use to produce at facility f. These resource bundles constitute a *resource game* (N, b^f_r) for each facility f and each resource r. Grouping the resources, one gets the function b^f. In the example presented in section 6.2, the resource games were additive : $b^f_r(S) = \sum_{i\in S} b^f_r(i)$ for all coalitions S.

The facilities are connected by three transport networks. These are represented by reflexive directed graphs, one for product transport denoted by D_P, one for resource transport denoted by D_R, and one for technology transport, denoted by D_T. We interpret these graphs as follows : $(f, f') \in D_R$, $(f, f') \in D_P$ or $(f, f') \in D_T$) denote that resources, products, or technology, respectively, can be transported from facility f to facility f'. We denote the set of all facilities to which a facility f can export resources, products or technology by $D_R(f)$, $D_P(f)$ or $D_T(f)$), respectively, and the set of all facilities from which a facility f can import resources, products or technology by $D_R^{-1}(f)$, $D_P^{-1}(f)$, $D_T^{-1}(f)$, respectively. The digraphs are assumed to be reflexive because at a facility f, the technology and resources of facility f itself are always available and products produced at f can always be sold at f.

In the example, there were neither transport costs nor license costs. In the general model, linear costs are associated to transport of resources and products. For $(f,f') \in D_R$, denote by $G_r^{ff'}$ the cost of transporting one unit of resource r from f to f'. Similarly, for $(f,f') \in D_P$ and $p \in P^f$ the cost of transporting one unit of product p from f to f' is denoted by $E_p^{ff'}$. In addition, assume that per unit of product $p \in P^g$ produced at facility $f \in D_T(g)$, a license fee L_p^{gf} has to be paid by the producer.

Finally, suppose not everything sent arrives, and denote by $\rho_r^{ff'}$ and $\pi_p^{ff'}$ the fraction which arrives after transport of resource r and product p from f to f', respectively.

A *controlled linear production situation with transport*, in short, an LPT, is a collection $\langle N,\ F,\ R,\ P,\ (P^f, A^f, c^f, b^f)_{f\in F},\ D_R,\ D_P,\ D_T, (E^{fg}, \pi^{fg})_{(f,g)\in D_P},\ (G^{fg}, \rho^{fg})_{(f,g)\in D_R},\ (L^{fg})_{(f,g)\in D_T}\rangle$ as described above.

With an LPT, we associate a transferable utility LPT-game (N, v) in the following way : N is the set of agents and the worth $v(S)$ of a coalition $S \in 2^N$ is the maximal value of a production plan using the resources S controls. More precisely, a production plan specifies which products are to be made where, according to which technology and with which resources. A production plan for coalition S has to satisfy the condition that at no facility more resources are used than the resources available after resource transport. After transport of the manufactured products to markets where they are most profitable, they are sold. The revenue obtained by this sale minus the costs generated by transport, is the value of the production plan.

Taking technology transport possibilities into account, one can see that at a facility f, every product p in

$$\bar{P}^f := \bigcup_{g\in D_T^{-1}(f)} P^g$$

can be produced. Again, successive transportation along more than one edge is not allowed. Hence we replace P^f with $\bar{P}^f$ and A^f with the matrix $\bar{A}$ formed by juxtaposition of the matrices in $\{A^g \mid g \in D_T^{-1}(f)\}$.

Moreover, suppose $(h,f) \in D_T$. Then each unit of product $p \in P^h \subseteq \bar{P}^f$ produced at facility f requires a license fee of L_p^{hf} to be paid and generates $c_p^g \pi_p^{fg} - E_p^{fg}$ when sold at a facility $g \in D_P(f)$. Denoting $y * z$ the vector with coordinates

$$(y * z)_k = y_k z_k$$

for two vectors y and z of the same size, we see that we can replace c^f with

$$\bar{c}^f := \max_{g\in D_P(f)} (c^g * \pi^{fg} - E^{fg}) - \sum_{h\in D_T^{-1}(f)} L^{hf} e^{P^h},$$

where e^{P^h} is the vector defined by

$$e_p^{P^h} = \begin{cases} 1 & \text{if } p \in P^h, \\ 0 & \text{if } p \notin P^h. \end{cases}$$

Hence, production of a bundle $\bar{q}^f \in \mathbf{R}^{\bar{P}^f}$ at a facility f requires the resource bundle $\bar{A}^f \bar{q}^f$ and yields a net payoff of

$$\langle \bar{c}^f, \bar{q}^f \rangle = \sum_{p \in \bar{P}^f} \bar{c}^f_p \bar{q}^f_p.$$

If we denote by t^{hf} the resource bundle transported from h to f, we see that the cost of this transport is $\langle G^{hf}, t^{hf} \rangle$ and that only the bundle $t^{hf} * \rho^{hf}$ arrives at f.

Hence, the worth of a coalition S is

$$\begin{aligned}
v(S) &= \max \sum_{f \in F} \left[\langle \bar{c}^f, \bar{q}^f \rangle - \sum_{h \in D_R^{-1}(f)} \langle G^{hf}, t^{hf} \rangle \right] \\
&\text{s. t.} \\
\bar{A}^f \bar{q}^f &\leq b^f(S) + \sum_{h \in D_R^{-1}(f)} (t^{hf} * \rho^{hf}) - \sum_{g \in D_R(f)} t^{fg} \text{ for all } f \in F, \\
\sum_{g \in D_R(f)} t^{fg} &\leq b^f(S) \text{ for all } f \in F, \\
\bar{q}^f &\geq 0 \text{ for all } f \in F, \\
t^{fg} &\geq 0 \text{ for all } (f,g) \in D_R.
\end{aligned}$$

Theorem 6.3.1 If all resource games (N, b^f_r) in a controlled linear production situation with transport are balanced, then the associated game (N, v) is also balanced and a core element can be computed by solving only one linear program, viz. the program which determines the worth of the grand coalition.

Proof : The linear program is bounded for each coalition S. This is ensured by the assumption that no product can be manufactured without using resources. Moreover, producing nothing is feasible for all S, so the programs are all feasible. Hence the dual programs are bounded and feasible, and the values of the primal and dual program coincide for each S. The dual program for coalition S is

$$\begin{aligned}
&\min \sum_{f \in F} (\langle y^f, b^f(S) \rangle + \langle z^f, b^f(S) \rangle) \\
&\text{s. t.} \\
y^f \bar{A}^f &\geq \bar{c}^f \text{ for all } f \in F, \\
y^f - y^g * \rho^{fg} + z^f &\geq -G^{fg} \text{ for all } (f,g) \in D_R, \\
y^f, z^f &\geq 0 \text{ for all } f \in F.
\end{aligned}$$

Note that y^f, $z^f \in \mathbf{R}^R$ for all $f \in F$ and that the feasible region of the dual program is independent of S. For each facility f and each resource r, (N, b^f_r) is balanced, hence take $u^f_r \in \mathbf{R}^N$ a core element of (N, b^f_r). Let $(y^f)_{f \in F}, (z^f)_{z \in F}$ be an optimal vector of the dual program of the grand coalition N. This vector is a feasible vector for the dual programs for all coalitions. Define $x \in \mathbf{R}^N$ by

$$x = \sum_{f \in F} \sum_{r \in R} (y^f_r + z^f_r) u^f_r.$$

Then for each coalition S,

$$\begin{aligned}\sum_{i\in S} x_i &= \sum_{f\in F}\sum_{r\in R}(y_r^f + z_r^f)\sum_{i\in S} u_{r,i}^f \\ &\geq \sum_{f\in F}\sum_{r\in R}(y_r^f + z_r^f) b_r^f(S) \\ &= \sum_{f\in F}(\langle y^f, b^f(S)\rangle + \langle z^f, b^f(S)\rangle) \\ &\geq v(S).\end{aligned}$$

The first inequality holds because u_r^f is a core element of (N, b_r^f), the second one because $(y^f)_{f\in F}, (z^f)_{z\in F}$ is a feasible vector for the dual program of S. If $S = N$ then these inequalities are equalities. Hence x is a core element of (N, v) and (N, v) is balanced. □

The optimal vector $(y^f)_{f\in F}, (z^f)_{z\in F}$ can be given an economic interpretation as follows. y_r^f is a shadow price for resource r when used at facility f. The constraints of the dual programs imply that $y_r^f + z_r^f \geq y_r^g \rho_r^{fg} - G_r^{fg}$ for each $(f,g) \in D_R$ and each resource r. The right hand side can be seen as the shadow price for resource r when transported from facility f to facility g. By complementary slackness, $y_r^f + z_r^f = y_r^g \rho_r^{fg} - G_r^{fg}$ if $t_r^{fg} > 0$. So, if any amount of resource r is exported from facility f, then $y_r^f + z_r^f$ is the maximal shadow price for resource r among the facilities $g \in D_R(f)$ to which resources can be transported from facility f. Hence z_r^f is the extra value of resource r when resource transport is allowed.

The converse of the theorem is not true in general, but for any LPT in which there exist a facility $f \in F$ and a resource $r \in R$ such that the resource game (N, b_r^f) is not balanced, alternative production matrices $(A^f)_{f\in F}$ and cost vectors $(c^f)_{f\in F}$ can be constructed such that the LPT-game associated to the modified LPT with these alternative production matrices and cost vectors is not balanced.

The players control the resources by means of the resource games. In theorem 6.3.1, these are required to be balanced and they are naturally non-negative. In Derks (1987) and Derks (1991), it is proved that this is equivalent to veto control.

Theorem 6.3.2 (Derks (1991), Theorem 2.8) A non-negative game is balanced if and only if it can be written as a non-negative linear combination of veto-rich control games.

Hence, the resource games $(b_r^f)_{f\in F, r\in R}$ are balanced if and only if the resources are veto-controlled, i.e. for each facility f, there exist a number k_f of bundles of resources $x_1^f, \ldots, x_{k_f}^f \in \mathbf{R}_+^R$ and veto-rich control games $w_1^f, \ldots, w_{k_f}^f$ such that

$$b^f(S) = \sum_{l=1}^{k_f} w_l^f(S) x_l^f \qquad \text{for all coalitions } S.$$

This implies that theorem 6.3.1 can be restated as

Theorem 6.3.3 The LPT-game associated to a controlled linear production situation with transport with veto-controlled resource bundles is balanced and a core element can be computed by solving just one linear program, viz. the program which determines the worth of the grand coalition.

Chapter 7

Controlled communication networks

In cooperative game theory, it is generally assumed that each subgroup of players can form and cooperate to obtain its value and that the grand coalition forms. Aumann and Dreze (1974) varied this assumption and introduced cooperative games with coalition structures in which a partition is assumed to form and within a partition element players cooperate.

However, this approach fails to take into account communication restrictions that may cause deficiencies in cooperation in some coalitions. Myerson (1977) introduced communication graphs to model non-transitive communication restrictions. In such a graph the vertices are the players and an edge between two players represents the fact that these players can communicate directly. The general procedure is the following : given a TU-game (N, v) and a communication graph $\langle N, E\rangle$, one defines a reward function r on collections of vertices and edges which takes the communication restrictions into account, and then new games are extracted.

This model was elaborated further by Owen (1986), Borm, Owen, and Tijs (1992), Van den Nouweland, Borm, Owen, and Tijs (1993), Van den Nouweland and Borm (1991) and Van den Nouweland (1993). A survey on this subject is given in Borm, Van den Nouweland, and Tijs (1994).

In this chapter, which is based on Feltkamp and Van den Nouweland (1993), we generalize communication situations by allowing the vertex set of the graph to differ from the set of players N, and by starting out with a reward function r, instead of deriving it. We assume that the players control the vertices and edges of the graph through so-called control games. In section 7.1 we provide the formal definition of controlled communication networks and we introduce three solution concepts for these networks, the Myerson value, the position value and the mixed value. These solution concepts are characterized axiomatically in section 7.2. Finally, in section 7.3 we present an method of constructing a TU-game corresponding to a controlled communication network, and we show that all TU-games can be obtained as such a network game.

7.1 One model, three solutions

Loosely speaking, a controlled communication network is a situation in which connected parts of a network produce revenue. Let us give an example.

Example 7.1.1 A sequencing situation $\langle N, (\alpha_i, s_i)_{i\in N}\rangle$ consists of a set $N = \{1, \ldots, n\}$ of players, who are waiting to be served by a counter in a specific order. Without loss of generality this order is assumed to be the standard order $sigma_0 = (1, \ldots, n)$. Each player places linear value on his time, i.e. if he has to wait for time t, his cost is $\alpha_i t$. We assume each player i has a serving time s_i.

With a sequencing situation $\langle N, (\alpha_i, s_i)_{i\in N}\rangle$, Curiel, Pederzoli, and Tijs (1989) associate a sequencing game (N, v), in which the worth $v(S)$ of a coalition S equals the maximal savings S can obtain by rearranging the members of S without exchanging the positions of players in S who have a player outside S between them.

Define a *connected coalition* to be a coalition of the form $\{k, k+1, \ldots, l\}$. For a connected coalition T, a T-permutation is a permutation σ of N which only permutes the members of T. The set of all T permutations is denoted $\Pi(T)$. For a permutation σ, denote the set $\{j \in N \mid \sigma(j) < \sigma(i)\}$ of predecessors of a player i under permutation σ by $P(i, \sigma)$. Then

$$v(T) = \max_{\sigma\in\Pi(T)} \sum_{i\in T} \alpha_i \sum_{k\in P(i,\sigma)} s_k - \sum_{i\in T} \alpha_i \sum_{k\in P(i,sigma_0)} s_k$$

for all connected coalitions T and defining for an arbitrary coalition S, $\mathcal{P}$ to be the coarsest partition of S into connected coalitions, we see that

$$v(S) = \sum_{T\in\mathcal{P}} v(T). \tag{7.1.1}$$

With a sequencing situation, one can define a graph $\langle V, E\rangle$ as follows. The set of vertices is defined by $V = N$ and the set of edges E consists of edges of the form $\{i, i+1\}$, with $i \leq n$. Then we see that the value of a coalition S is due to its connected components in the graph.

This way of presenting sequencing situations allows one to easily generalize the model, for example by dropping the assumption every person waiting to be served is an independent player. It could be that these persons are agents for companies, some agents being employed by the same company. In this case, the companies would be the players and the waiting costs of the agents of a company should be added, yielding the cost of waiting of the company.

We now formally introduce controlled communication networks. Consider a finite undirected graph $\langle V, E\rangle$ without loops or parallel edges. We assume that for each vertex $v \in V$ a veto-rich control game (N, c_v) is given and, similarly, for each edge $e \in E$ a veto-rich control game (N, c_e) is given. If c_v is the control game for vertex $v \in V$, then a coalition $S \subseteq N$ is allowed to use vertex v if and only if $c_v(S) = 1$. The control games (N, c_e) for edges have a similar interpretation.

Furthermore, we assume that there is a reward function r on subsets of vertices and edges $r : 2^V \times 2^E \to \mathbf{R}$, measuring the economic value of subnetworks. Keeping in mind that edges are to model communication channels, it seems reasonable to assume that an edge is useless without both its end points, i.e. for all $W \subseteq V$ and $F \subseteq E$ it holds that $r(W, F) = r(W, F \setminus \{\{v_1, v_2\}\})$ if $\{v_1, v_2\} \in F$ is such that $\{v_1, v_2\}$ is not a subset of W. So, the reward of a network $\langle W, F\rangle$ does not depend on the edges not in $F(W) := \{\{v, w\} \in F \mid v \in W, w \in W\}$. Moreover, in a network $\langle W, F\rangle$, W is partitioned into *communication components* in the following way : $C \subseteq W$ is a component within $\langle W, F\rangle$ if and only if $\langle C, F(C)\rangle$ is a connected subgraph of $\langle W, F(W)\rangle$ and is maximal with respect to this property. The resulting partition of W is denoted W/F. Correspondingly, we assume that the reward function is additive with respect to these components, i.e.

$$r(W, F) = \sum_{C \in W/F} r(C, F(C))$$

for all $W \subseteq V$ and all $F \subseteq E$.
For simplicity, we assume that r is *zero-normalized*, i.e. $r(\{v\}, \emptyset) = 0$ for all $v \in V$.

A *controlled communication network* is a 6-tuple $\langle N, V, E, (c_v)_{v\in V}, (c_e)_{e\in E}, r\rangle$ as described above. The set of all controlled communication networks with player set N will be denoted CCN^N.

Myerson (1977) and Borm, Owen, and Tijs (1992) consider reward functions of the special form

$$r^v(W, F) := \sum_{C \in W/F} v(C)$$

generated by a zero-normalized TU-game (N, v) and a graph $\langle N, E\rangle$, and present solutions of what they call the *communication situation* $\langle N, v, E\rangle$.

Here we present and axiomatically characterize solutions to the problem of distributing the reward $r(V, E)$ among the players in N. Formally, a *solution concept* on CCN^N is a function $\gamma : CCN^N \to \mathbf{R}^N$ assigning $\gamma_i(\mathcal{C})$ to player i in the controlled communication network $\mathcal{C} \in CCN^N$. One way to obtain solution concepts on CCN^N is to construct for each controlled communication network a TU-game corresponding to this network in which the players are the edges and/or vertices of the graph. To this game one can apply a solution concept from cooperative game theory, for example the Shapley value ϕ. This yields a value for edges and/or vertices and this value can be distributed among the players according to veto control. Concentrating on vertices this procedure yields the Myerson value (cf. Myerson, 1977) and concentrating on edges it yields the position value (cf. Borm, Owen, and Tijs, 1992). If no such distinction between vertices and edges is made, one obtains the mixed value.

Let $\mathcal{C} = \langle N, V, E, (c_v)_{v\in V}, (c_e)_{e\in E}, r\rangle$ be a controlled communication network. The *Myerson value* $\mu(\mathcal{C}) \in \mathbb{R}^N$ is defined by

$$\mu_i(\mathcal{C}) := \sum_{v\in V: i\in \text{veto}(c_v)} \frac{\phi_v(V, r_E)}{|\,\text{veto}(c_v)|}$$

for all $i \in N$, where the *vertex game* (V, r_E) is a game in which the vertices are the players, defined by $r_E(W) = r(W, E)$ for all $W \subseteq V$. Further, the *position value* $\pi(\mathcal{C}) \in \mathbb{R}^N$ is defined by

$$\pi_i(\mathcal{C}) := \sum_{e\in E: i\in \text{veto}(c_e)} \frac{\phi_e(E, r_V)}{|\,\text{veto}(c_e)|}$$

for all $i \in N$, where the *edge game* (E, r_V) is a game in which the edges are the players, defined by $r_V(F) = r(V, F)$ for all $F \subseteq E$. Finally, the *mixed value* $\rho(\mathcal{C}) \in \mathbb{R}^N$ is defined by

$$\rho_i(\mathcal{C}) := \sum_{a\in V\cup E: i\in \text{veto}(c_a)} \frac{\phi_a(V\cup E, \tilde{r})}{|\,\text{veto}(c_a)|}$$

for all $i \in N$, where the game $(V \cup E, \tilde{r})$ is defined by $\tilde{r}(J) = r(J \cap V, J \cap E)$ for all $J \subseteq V \cup E$.

7.2 Axiomatic characterizations

In this section we provide axiomatic characterizations of the three solution concepts introduced in section 7.1. It will turn out that all three concepts can be characterized by four axioms, three of which are identical for all solution concepts.

For convenience's sake, if $\mathcal{C}$ is a controlled communication network and $F \subseteq E$ a set of edges we will use $\mathcal{C}^{-F}$ to denote the network where the edges in F have been omitted and where the reward function has been restricted accordingly. We now introduce a few properties.

A solution concept γ on CCN^N is called *efficient* if for each controlled communication network $\mathcal{C}$, γ distributes exactly $r(V, E)$ among the players. In formula :

$$\sum_{i\in N} \gamma_i(\mathcal{C}) = r(V, E).$$

A solution concept γ on CCN^N is called *additive* if it is additive with respect to the reward function (ceteris paribus).

A solution concept γ on CCN^N is said to have the *superfluous edge property* if for all $\mathcal{C} \in CCN^N$ and all edges $e \in E$ that are superfluous for $\mathcal{C}$ it holds that

$$\gamma(\mathcal{C}) = \gamma(\mathcal{C}^{-\{e\}}).$$

Here, an edge $e \in E$ is called *superfluous for $\mathcal{C}$* if for all $F \subseteq E$,

$$r(V, F) = r(V, F \setminus \{e\}).$$

Note that for an edge e to be superfluous, we only consider the set V of all vertices; we do not demand $r(W, F) = r(W, F \setminus \{e\})$ for all $W \subseteq V$ and $F \subseteq E$. However, this turns out to be an equivalent requirement.

Lemma 7.2.1 Let $\mathcal{C} \in CCN^N$. Then an edge $e \in E$ is superfluous for $\mathcal{C}$ if and only if

$$r(W, F) = r(W, F \setminus \{e\})$$

for all $W \subseteq V$ and all $F \subseteq E$.

Proof : The "if" part is straightforward. For the "only if" part, note that

$$\begin{aligned} r(W, F) &= \sum_{C \in W/F} r(C, F(C)) \\ &= \sum_{C \in W/F(W)} r(C, F(C)) \\ &\overset{(*)}{=} \sum_{C \in V/F(W)} r(C, F(C)) = r(V, F(W)) \end{aligned}$$

for all $F \subseteq E$ and $W \subseteq V$.
Here, equality $(*)$ follows from the fact that r is zero-normalized.
Hence, for a superfluous edge e

$$\begin{aligned} r(W, F \setminus \{e\}) &= r(V, (F \setminus \{e\})(W)) \\ &= r(V, F(W) \setminus \{e\}) \\ &= r(V, F(W)) \\ &= r(W, F) \end{aligned}$$

for all $F \subseteq E$ and $W \subseteq V$. This competes the proof. □

In a graph $\langle V, E \rangle$, we denote by $D(V, E)$ the set of vertices that have degree at least 1, i.e. they have at least one neighbor in the graph, and we will shorten this notation to D whenever this does not lead to confusion.

The fourth property of solution concepts on CCN^N we introduce is anonymity. A controlled communication network is said to be *anonymous* if the reward function only depends on the number of edges and non-isolated vertices, i.e. there exists a function $f : \{0, \ldots, |D \cup E|\} \to \mathbb{R}$ such that

$$r(W, F) = f(|(W \cap D) \cup F|)$$

for all $W \subseteq V$ and $F \subseteq E$. A solution concept γ on CCN^N satisfies *anonymity* if for all anonymous $\mathcal{C} \in CCN^N$, the solution is proportional to the veto power of the players over the edges and non-isolated vertices or, in formula : there exists an $\alpha \in \mathbf{R}$ such that for all $i \in N$

$$\gamma_i(\mathcal{C}) = \alpha \cdot \sum_{a \in D \cup E : i \in \text{veto}(c_a)} \frac{1}{|\,\text{veto}(c_a)|}$$

The mixed value ρ satisfies the four properties mentioned. This is shown in

Lemma 7.2.2 The mixed value ρ satisfies efficiency, additivity, anonymity and the superfluous edge property.

Proof : Let $\mathcal{C}$ be a controlled communication network. Then

$$\begin{aligned}
\sum_{i \in N} \rho_i(\mathcal{C}) &= \sum_{i \in N} \sum_{a \in V \cup E : i \in \text{veto}(c_a)} \frac{\phi_a(V \cup E, \tilde{r})}{|\,\text{veto}(c_a)|} \\
&= \sum_{a \in V \cup E} \phi_a(V \cup E, \tilde{r}) \cdot \sum_{i \in \text{veto}(c_a)} \frac{1}{|\,\text{veto}(c_a)|} \\
&= \sum_{a \in V \cup E} \phi_a(V \cup E, \tilde{r}) \stackrel{(*)}{=} r(V, E),
\end{aligned}$$

where equality $(*)$ follows from efficiency of the Shapley value ϕ. Hence, ρ is efficient.

Additivity of ρ follows straightforwardly from additivity of ϕ.

In order to prove the superfluous edge property, take $\mathcal{C} \in CCN^N$ and $e \in E$ that is superfluous for $\mathcal{C}$. It clearly suffices to prove that $\phi_e(V \cup E, \tilde{r}) = 0$ and $\phi_a(V \cup E, \tilde{r}) = \phi_a(V \cup E \setminus \{e\}, \tilde{r})$ for all $a \in V \cup E \setminus \{e\}$. Using Lemma 7.2.1 we easily obtain $\tilde{r}(J) = \tilde{r}(J \setminus \{e\})$ for any $J \subseteq V \cup E$. Hence, e is a zero player in the game $(V \cup E, \tilde{r})$, and consequently $\phi_e(V \cup E, \tilde{r}) = 0$ and $\phi_a(V \cup E, \tilde{r}) = \phi_a(V \cup E \setminus \{e\}, \tilde{r})$ for all $a \in V \cup E \setminus \{e\}$. We conclude that ρ satisfies the superfluous edge property.

Now let $\mathcal{C} \in CCN^N$ be anonymous and let $f : \{0, \ldots, |D \cup E|\} \to \mathbb{R}$ be such that $r(W, F) = f(|(D \cap W) \cup F|)$ for all $F \subseteq E$ and $W \subseteq V$. Then all vertices $v \in V \setminus D$ are zero players in the game $(V \cup E, \tilde{r})$ and all $a \in D \cup E$ are symmetric in this game. By symmetry, efficiency, and the dummy property of ϕ this implies

$$\phi_a(V \cup E, \tilde{r}) = \begin{cases} \frac{f(|D \cup E|)}{|D \cup E|} & \text{if } a \in D \cup E \\ 0 & \text{if } a \in V \setminus D. \end{cases}$$

Hence, $\rho_a(\mathcal{C}) = \alpha \cdot \sum_{a \in D \cup E : i \in \text{veto}(c_a)} \frac{1}{|\,\text{veto}(c_a)|}$, where $\alpha := \frac{f(|D \cup E|)}{|D \cup E|}$. □

Before we prove that ρ is characterized by the four properties, we introduce two more definitions. Let $\langle V, E \rangle$ be a graph. Then we denote by $\mathcal{R}(V, E)$ the set of $\langle V, E \rangle$-admissible reward functions, i.e.

$$\mathcal{R}(V, E) := \left\{ r : 2^V \times 2^E \to \mathbf{R} \mid \begin{array}{c} r \text{ is additive w.r.t. components} \\ \text{and zero-normalized} \end{array} \right\}.$$

Moreover we denote the basic $\langle V, E\rangle$-admissible reward functions by

$$\mathcal{B}(V,E) := \left\{ u_{W,F} : 2^V \times 2^E \to \mathbb{R} \;\middle|\; \begin{array}{l} \langle W,F\rangle \text{ is a connected subgraph} \\ \text{of } \langle V,E\rangle \text{ and } |W| \geq 2 \end{array} \right\},$$

where $u_{W,F}$ is defined by

$$u_{W,F}(W',F') := \begin{cases} 1 & \text{if } W \subseteq W' \text{ and } F \subseteq F' \\ 0 & \text{otherwise.} \end{cases}$$

One can now prove the following result.

Lemma 7.2.3 Let $\langle V, E\rangle$ be a fixed graph. Then $\mathcal{B}(V,E)$ forms a basis of the vector space $\mathcal{R}(V,E)$.

This lemma is a straightforward corollary of the fact that the set $\{(N, u_S) \mid S \in 2^N \setminus \{\emptyset\}\}$ of unanimity games forms a basis of the class G^N of all TU-games and that the reward functions are zero-normalized. Though straightforward, the proof is rather technical and we therefore omit it.

Theorem 7.2.4 The mixed value ρ is the unique solution concept on CCN^N satisfying efficiency, additivity, anonymity, and the superfluous edge property.

Proof : According to Lemma 7.2.2 the mixed value ρ satisfies the four properties. Hence, we only have to show that there is at most one solution concept satisfying these properties. Suppose γ is a solution concept on CCN^N that satisfies the four properties. Using lemma 7.2.3 and additivity of γ and ρ, we see that it suffices to prove $\gamma = \rho$ for situations in which $r = \beta u_{W,F}$ for some $\beta \in \mathbb{R}$ and some connected subgraph $\langle W, F\rangle$ with $|W| \geq 2$. Hence, let $\mathcal{C} \subseteq CCN^N$ be a controlled communication network with $r = \beta u_{W,F}$ for some $\beta \in \mathbb{R}$ and some connected subgraph $\langle W, F\rangle$ with $|W| \geq 2$. Since every edge e in $E \setminus F$ is superfluous for $\mathcal{C}$, the superfluous edge property implies that $\gamma(\mathcal{C}) = \gamma(\mathcal{C}^{-E\setminus F})$. Furthermore,

$$r(W',F') = \beta u_{W,F}(W',F') = \begin{cases} \beta & \text{if } W \subseteq W' \text{ and } F' = F \\ 0 & \text{otherwise} \end{cases}$$

for all $W' \subseteq V$ and $F' \subseteq F$. Since $\langle W, F\rangle$ is a connected graph, it holds that $D(V,F) = W$. So, defining $f : \{0, \ldots, |D \cup E|\} \to \mathbb{R}$ by

$$f(k) = \begin{cases} \beta & \text{if } k = |W \cup F| \\ 0 & \text{otherwise} \end{cases}$$

we see that the controlled communication network $\mathcal{C}^{-E\setminus F}$ is anonymous. Now, by anonymity of γ we know that there exists an $\alpha \in \mathbb{R}$ such that

$$\gamma_i(\mathcal{C}^{-E\setminus F}) = \alpha \cdot \sum_{a \in W \cup F : i \in \mathrm{veto}(c_a)} \frac{1}{|\,\mathrm{veto}(c_a)|}$$

for all $i \in N$. Using efficiency, we obtain

$$\beta = r(V, F) = \sum_{i \in N} \gamma_i(\mathcal{C}^{-E \setminus F}) = \sum_{i \in N} \alpha \cdot \sum_{a \in W \cup F : i \in \text{veto}(c_a)} \frac{1}{|\,\text{veto}(c_a)|} = \alpha \cdot |W \cup F|.$$

Hence, $\alpha = \beta \cdot |W \cup F|^{-1}$ and recalling $\gamma(\mathcal{C}) = \gamma(\mathcal{C}^{-E \setminus F})$, we see

$$\gamma_i(\mathcal{C}) = \frac{\beta}{|W \cup F|} \cdot \sum_{a \in W \cup F : i \in \text{veto}(c_a)} \frac{1}{|\,\text{veto}(c_a)|} = \rho_i(\mathcal{C}).$$

□

We proceed by providing axiomatic characterizations of both the Myerson value and the position value. Both values can be characterized by efficiency, additivity, the superfluous edge property and an anonymity axiom. The anonymity axioms we need are *vertex anonymity* and *edge anonymity.*

A solution concept γ on CCN^N is said to be *vertex anonymous* if for every controlled communication network $\mathcal{C} \in CCN^N$ such that there exists a function $f : \{0, \ldots, |D(V, E)|\} \rightarrow \mathbb{R}$ with $r(W, E) = f(|D \cap W|)$ for all $W \subseteq V$, there is an $\alpha \in \mathbb{R}$ such that for all $i \in N$

$$\gamma_i(\mathcal{C}) = \alpha \cdot \sum_{v \in D : i \in \text{veto}(c_v)} |\,\text{veto}(c_v)|^{-1}.$$

A solution concept γ on CCN^N is called *edge anonymous* if for every $\mathcal{C} \in CCN^N$ such that there exists a function $f : \{0, \ldots, |E|\} \rightarrow \mathbb{R}$ with $r(V, F) = f(|F|)$ for all $F \subseteq E$, there is an $\alpha \in \mathbb{R}$ such that for all i in N

$$\gamma_i(\mathcal{C}) = \alpha \cdot \sum_{e \in E : i \in \text{veto}(c_e)} |\,\text{veto}(c_e)|^{-1}.$$

Theorem 7.2.5

i) The Myerson value μ is the unique solution concept on CCN^N that satisfies efficiency, additivity, the superfluous edge property and vertex anonymity.

ii) The position value π is the unique solution concept on CCN^N that satisfies efficiency, additivity, the superfluous edge property and edge anonymity.

The proof of theorem 7.2.5 runs along the same lines as the proof of lemma 7.2.2 and theorem 7.2.4 and therefore it is left to the reader.

Remark. The axiomatic characterizations of the Myerson value and the position value provided in theorem 7.2.5 are similar to axiomatic characterizations provided in Borm, Owen, and Tijs (1992) for communication situations. However, the reader should note that our characterizations hold for all controlled communication networks whereas Borm, Owen, and Tijs (1992) had to restrict to cycle-free communication graphs. The reason is that they were concerned with communication situations, in which a reward function is constructed from a game and a graph, and whereas they in fact used properties of the reward function, they defined the properties of a solution in function of properties of the game involved. Now some of these properties are not necessarily inherited by the reward function if the graph is not cycle-free.

7.3 Network games

In the previous sections we approached the problem dividing the reward $r(V, E)$ of a controlled communication network amongst the players in an indirect way, by first determining the value of vertices and edges and then distributing these values among the veto players in the corresponding control games. In this section we describe a direct way of dealing with the problem.

Let $\mathcal{C}$ be a controlled communication network. We define an associated game with player set N in the following way : for a coalition $S \subseteq N$, $V(S) := \{v \in V \mid c_v(S) = 1\}$ is the set of all *vertices* that coalition S can control and $E(S) := \{e \in E \mid c_e(S) = 1\}$ is the set of all *edges* that coalition S can control. Correspondingly, coalition S can obtain

$$v_{\mathcal{C}}(S) := r(V(S), E(S)).$$

Hence, we associate with $\mathcal{C} \in CCN^N$ the network game $(N, v_{\mathcal{C}})$ defined above. Subsequently, some solution concept for TU-games could be applied to the game $(N, v_{\mathcal{C}})$. We will pursue this line of research in the next chapter.

This approach seems interesting, because a number of games associated with economic situations can be seen to be network games in a more or less natural way. Some examples are sequencing games (Curiel, Pederzoli, and Tijs, 1989), permutation games (Tijs *et al.*, 1984), and assignment games (Shapley and Shubik, 1972).

Example 7.3.1 With a sequencing situation $\langle N, (\alpha_i, s_i)_{i \in N}\rangle$, associate a CCN situation $\mathcal{C} = \langle N, V, E, (c_v)_{v \in V}, (c_e)_{e \in E}, r\rangle$ by defining $V = N$, $E = \{\{i, i+1\} \mid i \in \{1, \ldots, n-1\}\}$, $c_i = u_i$, $c_{\{i,j\}} = u_{\{i,j\}}$ and

$$r(V, E) = \sum_{T \in V/E} v(T).$$

Then the network game $v_{\mathcal{C}}$ coincides with the sequencing game v, defined in equation 7.1.1.

However, some scepticism is in place here, because every TU-game is a network game in a trivial way : Let (N, v) be an arbitrary TU-game. Define a controlled communication network corresponding to (N, v) as follows. Let $V := N$ and $E := \{\{i, j\} \mid i, j \in N, i \neq j\}$. Hence, $\langle V, E\rangle$ is the complete graph with vertex set N. The control game for each vertex $i \in N$ is $(N, u_{\{i\}})$ and the control game for each edge $\{i, j\}$ is $(N, u_{\{i,j\}})$.

Hence, every player is a dictator for his own vertex and an edge between two vertices is controlled by the two players it connects. Finally, the reward function r assigns $v(S)$ to the subgraph $\langle S, E(S) \rangle$ for all $S \subseteq N$, and is extended in some feasible way to all subsets of vertices and edges. It is easily seen that the network game associated with the controlled communication network described above is the game (N, v).

Note that in the above discussion we did not restrict to zero-normalized games. However, the restrictions to zero-normalized reward functions was only made for simplicity and is not essential.

Chapter 8

Controlled economic situations

Both chapters 6 and 7 treated situations in which resources with which revenue can be generated are controlled by the players by means of control games. In this chapter, we present a unifying model of controlled economic situations. Associating a reward game to a controlled economic situation, we then study which properties of the reward function and the control games are inherited by the reward game.

In the preliminary section 8.1, we survey properties of control games and other simple games. Among other results it is shown that the only convex (resp. concave) control games are unanimity (resp. dual unanimity) games. Moreover, totally balanced simple games are shown to be precisely those simple games which have a population monotonic allocation scheme.

In section 8.2, we present controlled economic situations and use the results of section 8.1 to investigate the inheritance of properties of both the control games and the reward function by the reward game.

In section 8.3, a few examples of economic situations treated in the literature, such as flow situations and linear production situations will be reconsidered as controlled economic situations.

Finally, in section 8.4, we analyze controlled economic situations in which the set of resources is infinite and reconsider economies with land (Legut, Potters, and Tijs, 1994) as infinite controlled reward situations.

8.1 Simple games

In politicology and sociology, TU-games have been used to study various kinds of voting situations. There, typically, the worths of the coalitions are restricted to $\{0, 1\}$. The interpretation is that the coalitions S with worth 1 can decide collectively on the issue under consideration without the help of players outside S. Therefore, these coalitions are called *winning*. TU-games of this kind are called *simple games* and were considered first in von Neumann and Morgenstern (1944). Further studies on simple games are e.g.

Shapley-Shubik (1954), Shapley (1962), Banzhaf (1965), Shapley (1967), Dubey (1975), Dubey-Shapley (1979), Peleg (1981), Shapley(1981), Lehrer (1988) and Einy (1988).

In the literature, the discussion of simple games is mainly concentrated on monotonic simple games, based on the voting interpretation sketched above. However, if simple games are used to model not only theoretical power but also actual power, monotonicity may be lost. For example in parliament, a coalition which has the majority but which is composed of people with opposing interests might theoretically form a government, but internal conflict will prevent any bill being passed, while a subcoalition might succeed in passing bills.

In this section we survey properties of control games and simple games. Remember a *control game* (N, v) is a simple game which satisfies $v(N) = 1$. As a simple game v is completely determined by the set $W(v) := \{S \subseteq N \mid v(S) = 1\}$ of winning coalitions we will sometimes define a game by giving $W(v)$.

In this section, N will denote an arbitrary but fixed set of players and all games will have N as player set, unless specified otherwise. We often identify the game (N, v) with its characteristic function v. We denote the class of TU-games with player set N by G^N, the class of simple games with player set N by SG^N, and the class of control games with player set N by CG^N. For real numbers a and b, we denote $a \vee b := \max\{a, b\}$, and $a \wedge b := \min\{a, b\}$.

We first give some alternative formulations of monotonicity:

Lemma 8.1.1 Let $v \in G^N$ be a TU-game. The following four assertions are equivalent :

a. v is monotonic, i.e. $v(S) \leq v(T)$ if $S \subseteq T$.

b. $v(S \cup T) \geq v(S) \vee v(T)$ for all $S, T \subseteq N$.

c. $v(S \cup T) \geq v(S) \vee v(T)$ for all disjoint $S, T \subseteq N$.

d. $v(S \cap T) \leq v(S) \wedge v(T)$ for all $S, T \subseteq N$.

Proof : **a** implies **b** because $S \cup T$ contains both S and T.
c is a special case of **b**, so **b** implies **c**.
c $\Rightarrow$ **d** : Take $S, T \subseteq N$. Write $S = (S \cap T) \cup (S \setminus T)$, which is a disjoint union, and apply **c** : $v(S) \geq v(S \cap T) \vee v(S \setminus T) \geq v(S \cap T)$. Similarly, $v(T) \geq v(S \cap T)$, hence $v(S \cap T) \leq v(S) \wedge v(T)$.
d $\Rightarrow$ **a** : for any $S \subseteq T$, we have $S = S \cap T$, hence $v(S) = v(S \cap T) \leq v(S) \wedge v(T) \leq v(T)$.□

We now recall some further properties of simple games which have been used in the literature.

Definition 8.1.2 A simple game $v \in SG^N$ is *N-proper* if $v(S) \wedge v(N \setminus S) = 0$ for all $S \subseteq N$. It is *proper* if $v(S) \wedge v(T) = 0$ for all disjoint $S, T \subseteq N$. A simple game $v \in SG^N$ is *strongly proper* if $v(S) \wedge v(T) \leq v(S \cap T)$ for all $S, T \subseteq N$.

Note that simple proper games are N-proper, and simple monotonic N-proper games are proper. Recall that a game $v \in G^N$ is a *superadditive game* if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint S, $T \subseteq N$.

Lemma 8.1.3 Let $v \in SG^N$ be a simple game. Then

$$v \text{ is superadditive} \iff v \text{ is } N\text{-proper and monotonic}$$

Proof : Let v be a superadditive simple game. As v is non-negative (i.e. $v(S) \geq 0$ for all $S \subseteq N$), we have $v(S) \geq v(T) + v(S \setminus T) \geq v(T)$ for all $T \subseteq S \subseteq N$. Hence v is monotonic. Also, $1 \geq v(S \cup T) \geq v(S) + v(T)$ for any disjoint S and $T \subseteq N$, so at most one of $v(S)$ and $v(T)$ can equal 1. This yields $v(S) \wedge v(T) = 0$. Hence v is proper and so it is N-proper.

Conversely, suppose v is monotonic and N-proper. Then it is also proper, so for all disjoint S, $T \subseteq N$, we have $0 = v(S) \wedge v(T)$. Hence at least one of $v(S)$ and $v(T)$ equals zero, $v(S \cup T) \geq v(S) \vee v(T) = v(S) + v(T)$ and v is superadditive. □

A well-known result is that a control game v is balanced if and only if it is a *veto-rich game*, i.e. there is a player i such that $i \notin S$ implies $v(S) = 0$. Such a player is called a *veto player*. For a proof, we refer to Curiel (1988).

Theorem 8.1.4 (Curiel, 1988) A control game is balanced if and only if it has at least one veto player.

Corollary 8.1.5 A simple game is balanced if and only if it has a veto player and it is N-monotonic, i.e. if it is the zero game or it is a veto-rich control game.

Proof : If a simple game (N, v) is balanced, it is N-monotonic : for $x \in \text{Core}(v)$,

$$v(S) \leq x(S) \leq x(N) = v(N).$$

Now either $v(N) = 0$, in which case v is the zero game and every player is a veto player, or $v(N)$=1, and then it is a balanced control game and Curiel's theorem implies there is a veto player. Hence, in any case v has a veto player.

The converse implication is evident. □

Lemma 8.1.6 For a control game $v \in CG^N$ the following implications hold :

$$v \text{ is strongly proper} \Longrightarrow v \text{ is balanced} \Longrightarrow v \text{ is proper.}$$

Proof : Let v be strongly proper. It is balanced if and only if the set $\text{veto}(v) = \bigcap_{S \in W(v)} S$ of veto players is non-empty. Let S, $T \in W$. Then $v(S \cap T) \geq v(S) \wedge v(T) = 1$, hence $S \cap T \in W$. This means W is stable for finite intersection, and as W is finite, $\text{veto}(v)$ is winning. As the empty set is losing, $\text{veto}(v)$ is not empty, and v is balanced.

For the second implication, assume v to be balanced. A coalition S can only be winning if $\text{veto}(v) \subseteq S$, and for disjoint S, $T \subseteq N$, either $\text{veto}(v) \not\subseteq S$ or $\text{veto}(v) \not\subseteq T$, so either $v(S) = 0$ or $v(T) = 0$, hence $v(S) \wedge v(T) = 0$, and v is proper. □

Counterexamples for the converse implications in lemma 8.1.6 are given by the control games $(\{1,2,3\}, v_1)$ and $(\{1,2,3\}, v_2)$, defined by

$$\begin{aligned} W(v_1) &= \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}, \\ W(v_2) &= \{\{1,2\},\{1,3\},\{1,2,3\}\}. \end{aligned}$$

The game v_1 is (monotonic and) proper, but not balanced, and v_2 is (totally) balanced, but not strongly proper : $\text{veto}(v_2) = \{1\}$, and

$$0 = v_2(\{1,2\} \cap \{1,3\}) \not\geq v_2(1,2) \wedge v_2(1,3) = 1.$$

Furthermore, the first implication in lemma 8.1.6 does not hold for arbitrary simple games. This is shown by the simple game (N, v), defined by $N = \{1,2,3\}$ and $W(v) = \{\{1,2\}\}$. This game is strongly proper, but not a control game, nor balanced.

Lemma 8.1.7 For simple games, totally balancedness is equivalent to balancedness and monotonicity.

Proof : Let $v \in SG^N$ be a totally balanced. Obviously, v is balanced. Take $S \subseteq T \subseteq N$. We have to prove $v(S) \leq v(T)$. Because the game (T, v^T) is balanced, there exists a core element $x \in \text{Core}(v^T)$. So in particular, $x(T) = v(T)$, $x(S) \geq v(S)$, and $x_i \geq v(i) \geq 0$ for all $i \in T \setminus S$. This implies

$$v(T) = x(S) + x(T \setminus S) \geq v(S) + 0 = v(S).$$

Hence, v is monotonic.

For the reverse implication, suppose $v \in SG^N$ is balanced and monotonic, and let $T \subseteq N$. If $v(T) = 0$ then by monotonicity v^T is the zero game, which is balanced. If $v(T) = 1$, then any core element x of v satisfies $x_i \geq 0$ for all $i \in N \setminus T$, $x(T) \geq v(T) = 1$ and $x(N) = 1$. Hence, $x(T) = 1$. Also $x(S) \geq v(S) = v^T(S)$ for all $S \subseteq T$. Hence x restricted to T yields a core element of v^T, so v^T is balanced. □

Tijs (1981) introduced the τ-value. It is a one-point solution concept on the class of quasi-balanced games, which is larger than the class of balanced games.

Definition 8.1.8 (Tijs, 1981) For a TU-game $v \in G^N$, define the *upper vector* $M \in \mathbb{R}^N$ by $M_i = v(N) - v(N \setminus i)$ for all $i \in N$, and the *lower vector* $m \in \mathbb{R}^N$ by $m_i = \max_{S \subseteq N : i \in S} (v(S) - M(S \setminus \{i\}))$ for all $i \in N$. The game v is *quasi-balanced* if

$$m \leq M \qquad \text{and} \qquad m(N) \leq v(N) \leq M(N).$$

Note that for simple games both the lower and upper vector have only coordinates that are zero or one. This is easy to see for the upper vector, and for the lower vector it follows from the inequality $m_i \geq v(i)$.

Lemma 8.1.9 A simple monotonic quasi-balanced game is balanced.

Proof : Let v be monotonic, quasi-balanced and simple. If $v(N) = 0$, then v is the zero game, which is balanced. Otherwise, $M(N) \geq v(N) = 1$, hence there exists an $i \in N$ with $M_i = 1$, so with $v(N \setminus i) = 0$. By monotonicity, for all $S \subseteq N \setminus i$ we have $v(S) = 0$. Define x by $x_j = 0$ if $j \neq i$ and $x_i = 1$. Then $x(S) \geq v(S)$ for all coalitions S, and $x(N) = v(N)$. Hence v is balanced. □

A related solution concept was introduced by Sprumont (1990). He defined a population monotonic allocation scheme (PMAS) of a game (N, v) as a collection $x = \{x_{jS} \mid j \in S \subseteq N\}$ which satisfies the following two conditions

- $x_S(S) := \sum_{j \in S} x_{jS} = v(S)$ for all $S \subseteq N$.
- $x_{jS} \leq x_{jT}$ if $j \in S \subseteq T$.

Sprumont then proves that a TU-game (N, v) that has a PMAS x is totally balanced. For example, if x is a PMAS, then $x_S = (x_{iS})_{i \in S}$ is a core element of the subgame (S, v_S) for every coalition S.

The following theorem of Sprumont facilitates the task of identifying simple games which have a population monotonic allocation scheme.

Theorem 8.1.10 (Sprumont, 1990) A game $v \in G^N$ has a PMAS if and only if it is the sum of a positive linear combination of monotonic, balanced simple games and an additive game v_a given by $v_a(S) = a|S|$ for all $S \subseteq N$, where a is an arbitrary real number.

We now identify which simple games have a PMAS.

Theorem 8.1.11 For a simple game $v \in SG^N$ the following properties are equivalent :

- v has a PMAS.
- v is totally balanced,
- v is monotonic and balanced,
- v is monotonic and quasi-balanced,

Proof : Tijs (1981) proved balanced games are quasi-balanced. In view of the lemmas 8.1.7 and 8.1.9, this proves equivalence of the last three assertions. Equivalence of the first assertion with the second is a consequence of Sprumont's characterization of games having a PMAS : if a simple game is the sum of a positive linear combination of monotonic, balanced simple games and an additive game v_a, $a \in \mathbf{R}$, then the additive game is the zero game, and the positive linear combination of monotonic, balanced simple games is obviously monotonic and balanced. □

Without the monotonicity condition, the three varieties of balancedness are not equivalent.

Example 8.1.12 Consider $v \in SG^N$ with $N = \{1,2,3\}$ and $W(v) = \{\{1\},\{1,2,3\}\}$. This game is balanced but not totally balanced.

Example 8.1.13 Consider $v \in CG^N$ with $N = \{1,2,3,4\}$ and $W(v) = \{S \subseteq N \mid |S| = 2 \text{ or } |S| = 4\}$. This game is quasi-balanced : $M_i = 1$ for all $i \in N$, $m_i = 0$ for all $i \in N$. However, it is not proper since $v(12) \wedge v(34) = 1 \neq 0$. Hence using lemma 8.1.6, it is not balanced.

One could wonder whether properness of a simple game implies quasi-balancedness. That this is not the case is shown in the following example.

Example 8.1.14 Consider $v \in SG^N$ with $N = \{1,2,3\}$ and $v(S) = 1$ if $|S| \geq 2$. This proper and monotonic simple game is not quasi-balanced.

Furthermore, a quasi-balanced and proper control game is not necessarily balanced as is shown in the next example :

Example 8.1.15 Consider the control game $v \in CG^N$ defined by $N = \{1,2,3,4\}$ and $W(v) = \{\{1,2\},\{1,3\},\{2,3\},N\}$. Here $M_i = 1$ for all $i \in N$, $m_i = 0$ for all $i \in N$ and hence v is quasi-balanced. It is also proper, but it is not balanced. Suppose there existed an x in the core of v. Then $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$, $x_4 \geq 0$, adding these inequalities yields $2(x_1 + x_2 + x_3 + x_4) \geq 3$, a contradiction with efficiency : $x_1 + x_2 + x_3 + x_4 = 1$.

Definition 8.1.16 A TU-game $v \in G^N$ is *convex* if $v(S \cap T) + v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

Examples of convex games are the unanimity games u_S. In fact unanimity games are the only convex control games, as the following theorem shows.

Theorem 8.1.17 The following four assertions are equivalent for a game $v \in CG^N$:

a. v is convex

b. v is monotonic and strongly proper

c. $v(S \cap T) = v(S) \wedge v(T)$ for all $S, T \subseteq N$.

d. v is a unanimity game.

Proof : Using lemma 8.1.1 we see **b** and **c** are equivalent. We now prove that **a** implies **b**, **b** implies **d** and **d** implies **a**.

a $\Rightarrow$ **b** : a convex game v is superadditive, and a superadditive non-negative game is monotonic. In order to prove that v is strongly proper, take $S, T \subseteq N$. Now the inequality $v(S \cap T) \geq v(S) \wedge v(T)$ can only fail if $v(S) = v(T) = 1$. But then by convexity, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T) = 2$, and as all terms are either 0 or 1, both terms on the left-hand side have to equal 1, and in particular $v(S \cap T) = 1 \geq v(S) \wedge v(T)$. Hence v is strongly proper.

b $\Rightarrow$ **d** : the proof of lemma 8.1.6 yields that the set veto(v) of veto players is winning. By monotonicity, any coalition S containing veto(v) is winning. By definition, any winning coalition S contains veto(v). Hence, $v = u_{\text{veto}(v)}$.

d $\Rightarrow$ **a** : any unanimity game is convex. □

Definition 8.1.18 A TU-game v is *subadditive* if $v(S) + v(T) \geq v(S \cup T)$ for all disjoint $S, T \subseteq N$, and v is *concave* if $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

Shapley (1981) introduces strong monotonic simple games. We here call it N-strong. For monotonic simple games strongness and N-strongness coincide.

Definition 8.1.19 A simple game v is *N-strong* (cf. Shapley (1962)) if $v(S) \vee v(N \setminus S) \geq v(N)$ for all $S \subseteq N$. It is *strong* if $v(S) \vee v(T) \geq v(S \cup T)$ for all disjoint $S, T \subseteq N$.

The idea is that if one partitions a winning coalition of a strong game, at least one of the coalitions in the partition is winning.

Lemma 8.1.20 A simple game is strong if and only if it is subadditive.

Proof : Let v be a strong simple game, and let S, T be two disjoint subsets of N. Then $v(S) + v(T) \geq v(S) \vee v(T)$ because v is non-negative. Also $v(S) \vee v(T) \geq v(S \cup T)$. Combining the two inequalities we see v is subadditive.

Conversely let v be a subadditive simple game and let S, T be two disjoint subsets of N. Now $v(S) \vee v(T) \geq v(S \cup T)$ can fail only if $v(S) \vee v(T) = 0$. But this implies $0 = v(S) + v(T) \geq v(S \cup T)$, so the inequality holds after all. □

The study of strong, subadditive and concave simple games can be simplified by looking at the dual of a game :

Definition 8.1.21 The *dual* $v^* \in G^N$ of a TU-game $v \in G^N$ is defined by

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \subseteq N.$$

Clearly $v^{**} = v$ for all games $v \in G^N$. Some relations between a control game and its dual are given in

Lemma 8.1.22 For a control game v the following hold :

a. v^* is a control game,

b. v is monotonic iff v^* is monotonic,

c. v is N-proper iff v^* is N-strong,

d. v is convex iff v^* is concave.

The proofs are left to the reader.

Because unanimity games are convex, their duals u_S^* are concave. Note that $u_S^*(T) = 1$ if and only if S and T have a nonempty intersection. In fact, these games u_S^* are the only concave control games, as shown by the next corollary of theorem 8.1.17 and lemma 8.1.22.

Corollary 8.1.23 For a control game v the following are equivalent :

a. v is concave.

b. v is monotonic and subadditive.

c. $v(S \cup T) = v(S) \vee v(T)$ for all $S, T \subseteq N$.

d. v is the dual of a unanimity game.

For each $i \in N$, the *dictator game* $u_{\{i\}}$ is both a unanimity game and the dual of a unanimity game. In fact, we have

Corollary 8.1.24 Any of the following are equivalent for a control game $v \in CG^N$:

a. v is a dictator game.

b. v is convex and concave.

c. v is superadditive and subadditive.

d. v is proper and strong.

Proof : That **a** $\Rightarrow$ **b** $\Rightarrow$ **c** $\Rightarrow$ **d** is evident in view of the previous lemmata. It remains to be shown that **d** $\Rightarrow$ **a** : suppose v is a strong and proper game. Then strongness implies

$$\begin{aligned} 1 &= v(N) \\ &\leq v(N \setminus \{1\}) \vee v(\{1\}) \\ &\vdots \\ &\leq \max_{i \in N} v(\{i\}). \end{aligned}$$

Hence there exists an $i \in N$ with $v(\{i\}) = 1$. For distinct i and $j \in N$, properness implies $0 \geq v(i) \wedge v(j)$. Hence there exists exactly one $i \in N$ such that $v(i) = 1$. By properness, $v(S) \wedge v(i) = 0$ for $S \subseteq N \setminus \{i\}$, so any coalition not containing i has worth zero. On the other hand, if a coalition S contains i, then its complement $N \setminus S$ does not and has worth zero. Then strongness implies $1 = v(N) \leq v(S) \vee v(N \setminus S) = v(S)$ and S has to be a winning coalition. So i is the dictator, and $v = u_{\{i\}}$. $\square$

8.2 Reward games

In this section, we present controlled economic situations, which unify the economic situations presented in chapters 6 and 7.

Consider a finite set A of resources and a reward function $r : 2^A \to \mathbf{R}_+$, which assigns to each collection of resources, the reward that can be attained by using these resources. Assume these resources are controlled by agents in N, a finite set of agents, by means of control games, i.e. for each resource $a \in A$, there is a control game (N, c_a), which determines which coalitions may use the resource a as follows : S can use a if and only if $c_a(S) = 1$. With such a *controlled economic situation* $\langle N, A, (c_a)_{a\in A}, r\rangle$, one can associate a *reward game* (N, v_c^r), as follows :

$$v_c^r(S) := r(A(S)) \qquad \text{for all } S \subseteq N$$

where for a coalition S,

$$A(S) := \{a \in A \mid c_a(S) = 1\}$$

is the set of resources that coalition S can use. Note that $v_c^r(N) = r(A)$ because a control game c satisfies $c(N) = 1$. We will write v instead of v_c^r if this does not lead to confusion.

Note that (A, r) can be regarded as a game also, and when we require that the reward function r have a specific game-theoretic property, we mean that the game (A, r) should have this property.

In this section, we are interested in properties of the reward function and the control games that are inherited by the reward game. Hence, consider a fixed controlled economic situation $\langle N, A, (c_a)_{a\in A}, r\rangle$.

Theorem 8.2.1 If $x \in \text{Core}(r) \cap \mathbf{R}_+^A$ and $y^a \in \text{Core}(c_a)$ for all $a \in A$, then the vector

$$z := \sum_{a\in A} x_a y^a$$

is an element of the core of the reward game v.

Proof : The vector z is efficient :

$$\begin{aligned}\sum_{i\in N} z_i &= \sum_{i\in N}\sum_{a\in A} x_a y_i^a \\ &= \sum_{a\in A} x_a \\ &= r(A) \\ &= v(N)\end{aligned}$$

and z is coalitionally rational :

$$\sum_{i\in S} z_i = \sum_{i\in S}\sum_{a\in A} x_a y_i^a$$

$$\begin{aligned}
&= \sum_{a\in A} x_a \sum_{i\in S} y_i^a \\
&\geq \sum_{a\in A} x_a c(S) \\
&= \sum_{a\in A(S)} x_a \\
&\geq r(A(S)) \\
&= v(S).
\end{aligned}$$

Hence, z lies in the core of v. □

For a collection $(y^a)_{a\in A}$ with $y^a \in \text{Core}(c_a)$ for all $a \in A$, denote by Y the matrix with columns $(y^a)_{a\in A}$. Then the vector z of theorem 8.2.1 satisfies

$$z = Y \cdot x.$$

Denoting $\prod_{a\in A} \text{Core}(c_a)$ the set of all matrices Y as defined above, enables to restate theorem 8.2.1 as

$$\text{Core}(v) \supseteq \left(\prod_{a\in A} \text{Core}(c_a)\right) \cdot \text{Core}(r).$$

Here and in the rest of this section, all vectors are column vectors.

The converse inclusion is not true in general :

Example 8.2.2 Consider the situation $\langle\{1,2\},\{a,b,c\},(c_a,c_b,c_c),r\rangle$ where $N = \{1,2\}$, $c_a = u_N^*$, $c_b = c_c = u_N$, and r is given by

$$r(B) = \begin{cases} 0 & \text{if } |B| \leq 1, \\ 1 & \text{if } |B| \geq 2. \end{cases}$$

Then $\text{Core}(c_a) = \emptyset = \text{Core}(r)$, but $v = u_N$, and hence its core is not empty.

Example 8.2.3 Consider the situation $\langle\{1,2\},\{a,b,c\},(c_a,c_b,c_c),r\rangle$ where $N = \{1,2\}$, $c_a = u_{\{1\}}$, $c_b = c_c = u_{\{2\}}$, and r is given by

$$r(B) = \begin{cases} 0 & \text{if } |B| \leq 1 \text{ or } B = \{b,c\}, \\ 1 & \text{if } |B| \geq 2 \text{ and } B \neq \{b,c\}. \end{cases}$$

Denoting by $M^\top$ the transpose of a matrix M, we see $\text{Core}(c_a) = \{(\ 1\ \ 0\ )^\top\}$, $\text{Core}(r) = \{(\ 1\ \ 0\ \ 0\ )^\top\}$, $\text{Core}(c_b) = \text{Core}(c_c) = \{(\ 0\ \ 1\ )^\top\}$, hence the only vector in the core of v we can compute using theorem 8.2.1 is $(\ 1\ \ 0\ )^\top$, which is not the only element of the core of $v = u_N$.

A corollary of theorem 8.2.1 is

Corollary 8.2.4 If r is non-negative and balanced and all control games c_a are balanced, then the reward game is balanced.

Proposition 8.2.5 Let r and c_a be monotonic for all $a \in A$. Then v is monotonic.

Proof : c_a monotonic for all a implies that $A(S) \subseteq A(T)$ if $S \subseteq T$. Hence, $v(S) = r(A(S)) \leq r(A(T)) = r(T)$ if $S \subseteq T$. □

Proposition 8.2.6 If r is superadditive, and the control games $(c_a)_{a \in A}$ are all dictator games, then v is superadditive.

Proof : Take two disjoint coalitions S and T. For any $a \in A$, there is a player $i_a \in N$, such that $c_a = \delta_{i_a}$. This i_a does not belong to the intersection of S and T, which is empty. So, $A(S) \cap A(T) = \emptyset$. Furthermore, $\max(\delta_{i_a}(S), \delta_{i_a}(T)) = \delta_{i_a}(S \cup T)$ for all $a \in A$, which implies that $A(S \cup T) = A(S) \cup A(T)$. Then

$$\begin{aligned} v(S \cup T) &= r(A(S \cup T)) \\ &= r(A(S) \cup A(T)) \\ &\geq r(A(S)) + r(A(T)) \\ &= v(S) + v(T), \end{aligned}$$

so v is superadditive. □

The requirements on the control games in proposition 8.2.6 are very strict but they can be relaxed if the reward function is monotonic.

Proposition 8.2.7 If the reward function r is superadditive and monotonic, and the control games are all superadditive, then the reward game v is superadditive.

Proof : The control games are all superadditive and nonnegative, so they are monotonic. Hence, $c(S \cup T) \geq c(S) \vee c(T)$ for all $a \in A$ and each disjoint pair of coalitions S and T. This implies $A(S \cup T) \supseteq A(S) \cup A(T)$.

Furthermore, superadditivity of the control games implies properness by lemma 8.1.3, so $c_a(T) \wedge c_a(S) = 0$ for each disjoint pair S and T and for each resource a. Hence, $A(S) \cap A(T) = \emptyset$ for each disjoint pair S and T. This implies

$$\begin{aligned} v(S) + v(T) &= r(A(S)) + r(A(T)) \\ &\leq r(A(S) \cup A(T)) \\ &\leq r(A(S \cup T)) \\ &= v(S \cup T), \end{aligned}$$

so v is superadditive. □

We give an example to show that the assumptions of proposition 8.2.6 and 8.2.7 cannot be dropped.

Example 8.2.8 Let $N = \{1,2\}$, $A = \{a\}$, $c_a = u_N$, $r(A) = -10$. Then the reward function and the control game are superadditive, but $v = -10u_N$ is not superadditive.

The necessity of superadditivity of (a part of) the reward function is obvious, but replacing the superadditivity of the control functions by the weaker assumption of monotonicity does not guarantee superadditivity.

Example 8.2.9 Let $N = \{1, 2\}$, $A = \{a\}$, $c_a = u_N^*$, $r(A) = 1$. Then the reward function is superadditive and the control game is monotonic, but $v = u_N^*$ is not superadditive.

Subadditivity of the reward function in a controlled economic situation carries over to the reward game under similar circumstances as superadditivity.

Proposition 8.2.10 If the reward function is subadditive, and the control games are all dictator games, then the reward game is subadditive.

The proof being similar to the proof of proposition 8.2.6 we omit it.

Proposition 8.2.11 If the reward function is subadditive and monotonic and the control games are all subadditive and monotonic (and hence concave) games, then the reward game is subadditive (and monotonic).

The proof uses lemma 8.1.20 and runs along the lines of the proof of proposition 8.2.7 so it is left to the reader.

Proposition 8.2.12 If the reward function is convex and all control games are dictator games, then the reward game is convex.

Proof : Because the control games are dictator games, it follows that $A(S) \cup A(T) = A(S \cup T)$ and $A(S) \cap A(T) = A(S \cap T)$ for all coalitions S and T. Hence,

$$\begin{aligned} v(S \cup T) + v(S \cap T) &= r(A(S \cup T)) + r(A(S \cap T)) \\ &= r(A(S) \cup A(T)) + r(A(S) \cap A(T)) \\ &\geq r(A(S)) + r(A(T)) \\ &= v(S) + v(T), \end{aligned}$$

hence v is convex. □

As before, the assumptions on the control games can be relaxed if the assumptions on the reward game are strengthened.

Proposition 8.2.13 If the reward function is convex and monotonic and the control games are convex, then the reward game is convex (and monotonic).

Proof : According to theorem 8.1.17, convex control games are monotonic. This explains why the reward game is monotonic. From lemma 8.1.1, we know that a monotonic control game c satisfies $c(S \cup T) \geq c(S) \vee c(T)$, and from theorem 8.1.17 we know that

$c(s \cap T) = c(S) \wedge c(T)$ for all pairs S, T of coalitions. Hence,

$$\begin{aligned} v(S \cup T) + v(S \cap T) &= r(A(S \cup T)) + r(A(S \cap T)) \\ &\geq r(A(S) \cup A(T)) + r(A(S) \cap A(T)) \\ &\geq r(A(S)) + r(A(T)) \\ &= v(S) + v(T), \end{aligned}$$

so v is convex. □

In order to prove that the assumption of monotonicity is needed, consider example 8.2.8 again. It satisfies all assumptions except monotonicity of the reward function, and the resulting reward game is not convex.

We now state inheritance results for concavity which are similar to those obtained with convexity.

Proposition 8.2.14 If the reward function is concave and all control games are dictator games, then the resulting reward game is concave.

The proof is analogous to the proof of proposition 8.2.12.

Proposition 8.2.15 If the reward function is concave and monotonic and the control games are concave, then the reward game is concave (and monotonic).

Proof : A concave control game c satisfies $c(S \cup T) = c(S) \vee c(T)$ for all S and T by corollary 8.1.23. Furthermore, it is monotonic, so by lemma 8.1.1, $c(S \cap T) \leq c(S) \wedge c(T)$. Hence, $A(S \cup T) = A(S) \cup A(T)$ and $A(S \cap T) \leq A(S) \cap A(T)$. It follows that

$$\begin{aligned} v(S \cup T) + v(S \cap T) &= r(A(S \cup T)) + r(A(S \cap T)) \\ &\leq r(A(S) \cup A(T)) + r(A(S) \cap A(T)) \\ &\leq r(A(S)) + r(A(T)) \\ &= v(S) + v(T), \end{aligned}$$

hence v is concave. □

Population monotonic allocation schemes carry over to the reward game also.

Theorem 8.2.16 Let r have a PMAS x of which all entries are non-negative, let each control game c_a have a PMAS y^a and assume all control games are monotonic. Then $z = (z_{i,S})_{i,S}$, defined by

$$z_{i,S} = \sum_{a \in A(S)} y^a_{i,S} x_{a,A(S)}$$

is a PMAS of the reward game.

Proof :

$$
\begin{aligned}
\sum_{i\in S} z_{i,S} &= \sum_{i\in S}\sum_{a\in A(S)} y^a_{i,S}x_{a,A(S)} &&= \sum_{a\in A(S)}\sum_{i\in S} y^a_{i,S}x_{a,A(S)} \\
&= \sum_{a\in A(S)} c_a(S)x_{a,A(S)} &&= \sum_{a\in A(S)} x_{a,A(S)} \\
&= r(A(S)) \\
&= v(S).
\end{aligned}
$$

Furthermore, for $i \in T \subseteq S$,

$$
\begin{aligned}
z_{i,T} &= \sum_{a\in A(T)} y^a_{i,T}x_{a,A(T)} &&\leq \sum_{a\in A(T)} y^a_{i,S}x_{a,A(T)} \\
&\leq \sum_{a\in A(T)} y^a_{i,S}x_{a,A(S)} &&\leq \sum_{a\in A(S)} y^a_{i,S}x_{a,A(S)} \\
&= z_{i,S}.
\end{aligned}
$$

The first inequality follows because y is a PMAS and x is non-negative, the second because y is non-negative and x is a PMAS, and the third because the control games are monotonic, hence $A(T) \subseteq A(S)$. □

8.3 Examples of controlled economic situations

In this section, we reformulate economic situations that have been studied in the literature and with which games have been associated. Theorems from section 8.2 can then be invoked to prove properties of these games.

Example 8.3.1 Kalai and Zemel (1982) considered flow situations, introduced flow games and proved using a theorem of Ford and Fulkerson (1954) that these are balanced. In these flow situations, vertices of a graph were (owned by) players. Curiel, Derks and Tijs (1989) generalized flow situations to flow situations with committee control and introduced associated flow games with committee control. A flow situation with committee control consists of a set V of nodes, among which a source s and a sink t, a directed graph $D \subseteq V \times V$, a set of users N, for each arc a in the graph a control game c_a with player set N and a non-negative capacity k_a. With each coalition $S \subseteq N$, we associate the graph $D(S) = \{a \in D \mid c_a(S) = 1\}$ controlled by S. Now the worth of a coalition S is defined as the maximal flow from the source to the sink through the graph $D(S)$. Curiel, Derks and Tijs then prove that a flow game with committee control is balanced if all control games are balanced.

Now define a reward function $r : D \to \mathbf{R}_+$ by putting $r(B)$ equal to the maximal flow through the graph B from source to sink, for all $B \subseteq D$. Considering (D, r) as a game, we see that it is is exactly a flow game as defined by Kalai and Zemel, hence the game (D, r) is balanced. Hence, using theorem 8.2.1, we see that the associated reward game is balanced. But the reward game assigns to every coalition S the maximal flow from source to sink through the graph controlled by S, and hence coincides with the flow game with committee control.

Example 8.3.2 Linear production situations with transport possibilities as described in chapter 6 can be modeled as controlled economic situations. Consider an LPT, and assume the resource games (N, b_r^f) are balanced. Using the remark after theorem 6.3.2, we see that for each facility f, there exist a number k^f of non-negative bundles of resources $x_1^f, \ldots, x_{k_f}^f \in \mathbf{R}_+^R$ and veto-rich control games $w_1^f, \ldots, w_{k_f}^f$ such that

$$b^f(S) = \sum_{l=1}^{k_f} w_l^f(S) x_l^f \qquad \text{for all coalitions } S.$$

Now define a controlled economic situation as follows. The set of resources equals $A := \{x_l^f \mid f \in F \text{ and } l = 1, \ldots, k_f\}$, the reward function r assigns to every collection $B \subseteq A$ of bundles the maximal value of a production plan using the bundle of resources $\sum_{x \in B} x$. The control game of a bundle x_l^f is the game w_l^f. Now (A, r) is a linear production game as modeled in Owen (1975b). Hence, it is balanced.

The associated reward game assigns to every coalition S the maximal value of a production plan using the resources which S controls. Hence, it coincides with the LPT-game. Using the fact that the control games are balanced and invoking theorem 8.2.1, we see that the LPT-game is balanced.

Other situations can be modeled as linear production situations. For example, pooling games (Potters and Tijs, 1987) are linear production games (Tijs, 1992) and hence are balanced.

Example 8.3.3 A controlled communication network $\langle N, V, E, (c_v)_{v \in V}, (c_e)_{e \in E}, r\rangle$ described in chapter 7 can be considered as a controlled economic situation by defining $A = V \cup E$ and taking $\tilde{r}$, defined by $\tilde{r}(B) = r(B \cap V, B \cap E)$ for all $B \subseteq A$ as reward function. The associated reward game coincides with the network game described in section 7.3.

Moreover, in a controlled communication network the control games are balanced and the reward function is assumed zero-normalized. Hence, the conditions of theorem 8.2.7 are partly satisfied and we have the following corollaries :

Corollary 8.3.4 If in a controlled communication situation the reward function is superadditive and the control games are monotonic then the network game is superadditive.

Proof : If the reward function is superadditive and zero-normalized it is monotonic. Furthermore, it follows from lemma 8.1.6 that a balanced and monotonic control game is superadditive. Hence, by theorem 8.2.7, the network game is superadditive. □

Corollary 8.3.5 If in a controlled communication network the reward function is balanced then the network game is balanced as well.

8.4 Infinite controlled economic situations

In this section, we turn our attention to infinite controlled economic situations. These are situations in which the set of resources is an infinite set.

Definition 8.4.1 An infinite controlled economic situation $\langle N, A, \mathcal{A}, c, r\rangle$ consists of the following. $(A, \mathcal{A})$ is a measure space, $r : \mathcal{A} \to \mathbf{R}_+$ is a set function satisfying $r(\emptyset) = 0$. As before, r is the reward function. N is a finite set of players and $c : A \to CG^N$ is a measurable function with respect to the measure spaces $(A, \mathcal{A})$ and $(CG^N, 2^{CG^N})$. The mapping c assigns to each element $a \in A$, a control game, denoted c_a, which specifies which coalitions can use a.

The associated reward game (N, v) is defined by putting $A(S) := \{a \in A \mid c_a(S) = 1\}$ for all $S \subseteq N$. Note that $A(S) = c^{-1}(\{v \in CG^N \mid v(S) = 1\})$ is measurable. Now define $v(S) = r(A(S))$.

$(A, \mathcal{A}, r)$ is an infinite game. Under similar conditions as in the finite case, a core element or a PMAS of the reward game and the control games induce a core element or a PMAS of the associated reward game. Fix an infinite controlled economic situation $\langle N, A, \mathcal{A}, c, r\rangle$.

Theorem 8.4.2 Suppose there exists a measure $\mu : \mathcal{A} \to \mathbf{R}_+$ with $\mu(B) \geq r(B)$ for all $B \in \mathcal{A}$ and $\mu(A) = r(A)$. Suppose the control games (N, c_a) are balanced for almost all $a \in A$ and there exists a measurable function $d : A \to \mathbf{R}^N$ satisfying

$$\begin{cases} \sum_{i\in S} d_i(a) & \geq & c_a(S) & \text{for all } S \subseteq N \text{ and almost everywhere,} \\ \sum_{i\in N} d_i(a) & = & c_a(N) & \text{almost everywhere.} \end{cases}$$

Then the reward game (N, v) is balanced.

Proof : Define $x := \int_A d(a)d\mu(a) \in \mathbf{R}^N$. Then for any coalition $S \subseteq N$,

$$\begin{array}{rcccc} \sum_{i\in S} x_i & = & \displaystyle\sum_{i\in S}\int_A d_i(a)d\mu(a) & = & \displaystyle\int_A \sum_{i\in S} d_i(a)d\mu(a) \\ & \geq & \displaystyle\int_A c_a(S)d\mu(a) & = & \displaystyle\int_{A(S)} 1 d\mu(a) \\ & = & \mu(A(S)) & \geq & r(A(S)) \\ & = & v(S). & & \end{array}$$

For $S = N$, the two inequalities are equalities, hence x is a core-element of the reward game. $\square$

Definition 8.4.3 A PMAS of an infinite game $(A, \mathcal{A}, r)$ is a collection $(\mu_B)_{B\in\mathcal{A}}$ of measures satisfying

- μ_B is a measure on the measure space $(B, \{D \cap B \mid D \in \mathcal{A}\})$.

- $\mu_B(B) = r(B)$ for all $B \in \mathcal{A}$.
- $\mu_B(D) \leq \mu_C(D)$ for all measurable $D \subseteq B \subseteq C$.

With this definition, one can prove a theorem similar to theorem 8.2.16.

Theorem 8.4.4 If the reward function has a non-negative PMAS $\mu = (\mu_B)_{B \in \mathcal{A}}$, if there exists a function f that assigns to every $a \in A$ a PMAS f^a and if the control games are monotonic for almost every $a \in A$ (w.r.t μ), then the reward game has a PMAS x, defined by

$$x_{i,S} = \int_{A(S)} f^a_{i,S} d\mu_{A(S)}(a).$$

The proof is similar to the proof of theorem 8.2.16 and it is left to the reader.

Examples of infinite controlled economic situations are the economies with land, as presented by Legut, Potters and Tijs.

Example 8.4.5 Legut, Potters and Tijs (1994) introduced land games associated with economies with land. These are defined as follows : there is an area of land A, to be divided among a set N of players. It is assumed there is a σ-algebra $(A, \mathcal{A})$ and the worth to a player i of parcels of land $B \in \mathcal{A}$ is given by a non-negative measure μ_i. Since the measures are absolutely continuous with respect to the measure $\nu = \sum_{i \in N} \mu_i$, we can write (by the theorem of Radon-Nikodym)

$$\mu_i(B) = \int_B f_i d\nu,$$

where $(f_i)_{i \in N}$ are bounded ν-measurable functions on A. The function f_i is called the utility density of agent i. Each agent i is endowed with an initial measurable parcel of land A_i, such that the collection $(A_i)_{i \in N}$ forms an $\mathcal{A}$-(measurable) partition of A. An economy with land is a triple

$$\mathcal{E} := \{N, \{A_i, f_i\}_{i \in N}\}$$

with N, A_i and f_i described above. The corresponding transferable utility land game is defined as follows : every measure μ_i is assumed to measure the monetary value of land for player i. The worth of a coalition S is given by

$$v_{\mathcal{E}}(S) = \sup\{\sum_{i \in S} \int_{Y_i} f_i d\nu \mid (Y_i)_{i \in S} \text{ is an } \mathcal{A}\text{-partition of } A(S) := \bigcup_{i \in S} A_i\}$$

for all $S \subseteq N$. Legut, Potters and Tijs prove that $v_{\mathcal{E}}(S) = \int_{A(S)} \bigvee_{i \in S} f_i \, d\nu$.

With an economy with land, we associate an infinite controlled economic situation with player set N and set of resources A. The measurable control function is given by $c_a = u_i$ if $a \in A_i$. The reward function r is defined by

$$r(B) = \int_B \bigvee_{i \in N(B)} f_i \, d\nu,$$

where $N(B) := \{i \in N \mid B \cap A_i \neq \emptyset\}$ contains all players owning a parcel of B.

Computing the associated reward game, we obtain

$$\begin{aligned} v_c^r(S) &= r(A(S)) \\ &= \int_{A(S)} \bigvee_{i \in N(A(S))} f_i \, d\nu \\ &= \int_{A(S)} \bigvee_{i \in S} f_i \, d\nu \\ &= v_{\mathcal{E}}(S) \end{aligned}$$

for all $S \subseteq N$. Hence, we can apply theorems 8.4.2 or 8.4.4 to verify whether the land game has a core element or a PMAS.

Chapter 9

TU-games, simple games and control games

Because traditionally simple games are used to model voting situations, a solution concept on the class of simple games is also called a power index : it measures the power of a voter. Shapley and Shubik (1954) introduced the Shapley-Shubik index, which is the Shapley value restricted to simple games. Dubey (1975) characterized this index axiomatically on the class of monotonic simple games. Another power index is the Banzhaf index, which was introduced by Banzhaf (1965) and which was characterized axiomatically by Dubey and Shapley (1979), again on the class of monotonic simple games. Einy (1988) extended these axiomatic characterizations to several classes of monotonic TU-games. The proofs of the characterizations on the class of monotonic simple games use *minimal* winning coalitions, i.e. winning coalitions such that every subcoalition is losing. While this concept is natural for monotonic simple games, it is not for non-monotonic simple games.

In this chapter, which is based on Feltkamp (1995), a different line of proof shows that with axioms similar to those of Dubey (1975), one can characterize the Shapley value on the class of control games, the class of all simple games, and also on the class of all TU-games. With a different efficiency axiom, we also extend the characterization of the Banzhaf value tc these classes.

9.1 Axiomatic characterizations of the Shapley and Banzhaf values

We start out by recalling some definitions and notations used in this chapter.

In the sequel, N will denote an arbitrary but fixed set of players and all games will have N as player set, unless specified otherwise. As before, we denote the class of TU-games with player set N by G^N, the class of simple games with player set N by SG^N, and the class of control games with player set N by CG^N.

For real numbers a and b, we denote $a \vee b := \max\{a, b\}$, and $a \wedge b := \min\{a, b\}$. For TU-games v, $w \in G^N$, $v \vee w$ and $v \wedge w$ denote the games defined by

$$(v \vee w)(S) := v(S) \vee w(S) \qquad \text{for all } S \subseteq N$$

$$(v \wedge w)(S) := v(S) \wedge w(S) \qquad \text{for all } S \subseteq N.$$

For each of the classes of simple games, control games and monotonic simple games it holds that if v and w are in the class, so are $v \vee w$ and $v \wedge w$.

A *solution concept* or *value* on a class $\mathcal{C}^N \subseteq G^N$ of TU-games is a vector valued function $\psi : \mathcal{C}^N \to \mathbb{R}^N$, assigning the real number $\psi_i(v)$ to each player i in the game $v \in \mathcal{C}^N$.

We proceed by providing some properties of a solution concept on a class $\mathcal{C}^N$.

- A solution ψ is *efficient* if $\sum_{i \in N} \psi_i(v) = v(N)$ for all games $v \in \mathcal{C}^N$.

- A solution ψ is *anonymous* if for all $v \in \mathcal{C}^N$ and for all permutations σ of N such that $\sigma v \in \mathcal{C}^N$,

$$\psi_{\sigma(i)}(v) = \psi_i(\sigma v) \qquad \text{for all } i \in N,$$

  where the game σv is defined by

$$\sigma v(S) = v(\sigma(S)) \qquad \text{for all } S \subseteq N.$$

- A *null player* in a game $v \in \mathcal{C}^N$ is a player $i \in N$ such that $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$ containing i.

  A solution ψ has the *null player property* if $\psi_i(v) = 0$ for all games $v \in \mathcal{C}^N$ with null player i.

- A *carrier* of a game $v \in \mathcal{C}^N$ is a coalition $T \subseteq N$ such that $v(S) = v(S \cap T)$ for all $S \subseteq N$.

  A solution ψ has the *carrier property* if $\sum_{i \in T} \psi_i(v) = v(T)$ for all games $v \in \mathcal{C}^N$ and each carrier T of v.

- A solution ψ has the *transfer* property if

$$\psi(v \vee w) + \psi(v \wedge w) = \psi(v) + \psi(w)$$

  for all games v, $w \in \mathcal{C}^N$ such that $v \vee w$, $v \wedge w \in \mathcal{C}^N$.

- A solution is *additive* if

$$\psi(v + w) = \psi(v) + \psi(w)$$

  for all games v, $w \in \mathcal{C}^N$ such that $v + w \in \mathcal{C}^N$.

The following should be noted : if $\mathcal{C}_1^N \subseteq \mathcal{C}_2^N$ and a solution ψ satisfies any of the properties described above on $\mathcal{C}_2^N$, it satisfies the property on the class $\mathcal{C}_1^N$ as well. On the class of control games, the additivity property is useless : all control games have a winning grand coalition, hence the sum of two control games is not a control game.

Furthermore, a value which is additive on the class G^N of all TU-games satisfies the transfer property on G^N and hence also on any subclass. To prove this, take $v, w \in G^N$. Then, using additivity,

$$\begin{aligned}\psi(v \vee w) + \psi(v \wedge w) &= \psi(v \vee w + v \wedge w)\\ &= \psi(v + w)\\ &= \psi(v) + \psi(w).\end{aligned}$$

Finally, we note that the carrier property is equivalent to the efficiency and null player properties together.

A widely studied solution concept is the Shapley value ϕ of a TU-game $v \in G^N$, defined by

$$\phi_i(v) = \sum_{S:i\in S} \frac{|N \setminus S|!\,|S \setminus \{i\}|!}{|N|!}(v(S) - v(S \setminus \{i\}))$$

for all $i \in N$.

It is well known that the Shapley value is efficient, anonymous, additive and satisfies the null player property on G^N and hence on any subclass of G^N. The remark above shows that it satisfies the transfer property on any class of TU-games.

The following theorem is analogous to *Theorem* II in Dubey (1975).

Theorem 9.1.1 The unique value on the class CG^N of control games satisfying efficiency, anonymity, the null player property and the transfer property is the Shapley value.

Proof : It is clear that the Shapley value satisfies the four properties mentioned in the theorem. Suppose a solution concept ψ satisfies these four properties as well. We prove ψ coincides with the Shapley value ϕ.

First, Dubey (1975) proved that the Shapley value is the unique value on the class of monotonic simple games satisfying anonymity and the carrier and transfer properties. The carrier property is equivalent to efficiency and the null player property combined, hence ψ coincides with the Shapley value on this class.

In order to extend this result to the class of all control games, we introduce the Dirac games δ_S defined by

$$\delta_S(T) = \begin{cases} 1 & \text{if } T = S,\\ 0 & \text{if } T \neq S,\end{cases}$$

for all $S \subseteq N$. For $S \subset N$, let the control games δ'_S be defined by $\delta'_S = \delta_S + \delta_N$. Note that $(u_S - \delta_S) \vee \delta'_S = u_S$ and $(u_S - \delta_S) \wedge \delta'_S = \delta_N = u_N$ for all $S \subset N$. Using the transfer property and the fact that $u_S - \delta_S$ is a control game we obtain

$$\psi(u_N) + \psi(u_S) = \psi(u_S - \delta_S) + \psi(\delta'_S).$$

$$\text{Hence } \psi(\delta'_S) \begin{aligned}[t] &= \psi(u_N) + \psi(u_S) - \psi(u_S - \delta_S) \\ &\overset{(1)}{=} \phi(u_N) + \phi(u_S) - \phi(u_S - \delta_S) \\ &\overset{(2)}{=} \phi(\delta'_S), \end{aligned}$$

where (1) follows from the monotonicity of u_N, u_S, $u_S - \delta_S$ and coincidence of ψ and ϕ on the class of monotonic simple games, and (2) because the Shapley value ϕ satisfies the transfer property.

Note that any arbitrary control game v can be written

$$v = \bigvee_{T \in W(v)} \delta'_T.$$

We prove $\psi(v) = \phi(v)$ for all $v \in CG^N$ by induction on $|W(v)|$:

- if $|W(v)| = 1$, then $v = u_N$ which is monotonic, hence $\psi(v) = \phi(v)$.
- if $|W(v)| = 2$, then $v = \delta'_T$ for some $T \subset N$, hence $\psi(v) = \phi(v)$.
- Choose $k \geq 2$ and suppose $\psi(v)$ coincides with $\phi(v)$ on all games $v \in CG^N$ with $|W(v)| \leq k$. Take a game v with $|W(v)| = k + 1$, and choose a $T \in W(v) \setminus \{N\}$. Then $v = (v - \delta_T) \vee \delta'_T$, $(v - \delta_T) \wedge \delta'_T = u_N$ and $W(v - \delta_T) = W(v) \setminus \{T\}$, so $|W(v - \delta_T)| = k$. Hence by the transfer property and the induction hypothesis

$$\begin{aligned} \psi(v) &= \psi(v - \delta_T) + \psi(\delta'_T) - \psi(u_N) \\ &= \phi(v - \delta_T) + \phi(\delta'_T) - \phi(u_N) \\ &= \phi(v). \end{aligned}$$

This proves the uniqueness of a solution satisfying the four properties on CG^N. □

Along the same lines one can prove

Theorem 9.1.2 The unique value on the class SG^N of simple games satisfying efficiency, anonymity, the null player property and the transfer property is the Shapley value.

In order to characterize the Shapley value on the class of all TU-games, we first need some lemmata. The zero game in G^N is denoted by $\underline{0}$.

Lemma 9.1.3 Let ψ be a solution on G^N satisfying the transfer property, with $\psi(\underline{0}) = 0$. Then for all games $v \in G^N$,

$$\psi(v) = \sum_{S \subseteq N} \psi(v(S)\delta_S). \tag{9.1.1}$$

Proof : We prove in three steps that equation (9.1.1) holds.

1. For the class of all non-negative games v the proof is by induction on

$$k(v) := |\{S \subseteq N \mid v(S) > 0\}|.$$

(A game v is non-negative if $v(S) \geq 0$ for all $S \subseteq N$.)

- If $k(v) = 0$ then $v = \underline{0}$ and so $\psi(v) = 0 = \sum_{S \subseteq N} \psi(v(S)\delta_S)$.
- Take $k > 0$ and suppose equation (9.1.1) holds for all non-negative games v with $k(v) < k$. For a non-negative game v with $k(v) = k$, choose a coalition $T \subseteq N$ such that $v(T) > 0$. Then $k(v - v(T)\delta_T) = k - 1$, $(v - v(T)\delta_T) \vee (v(T)\delta_T) = v$ and $(v - v(T)\delta_T) \wedge (v(T)\delta_T) = 0$, hence using the induction hypothesis and the transfer property, we obtain

$$\begin{aligned}\psi(v) &= \psi[v - v(T)\delta_T] + \psi[v(T)\delta_T] - \psi[(v - v(T)\delta_T) \wedge v(T)\delta_T] \\ &= \sum_{S \subseteq N} \psi[(v - v(T)\delta_T)(S)\,\delta_S] + \psi[v(T)\delta_T] - \psi(\underline{0}) \\ &= \sum_{S \in 2^N \setminus \{T\}} \psi(v(S)\delta_S) + \psi(v(T)\delta_T) \\ &= \sum_{S \subseteq N} \psi(v(S)\delta_S).\end{aligned}$$

2. For non-positive games one proves analogously (interchanging the operations $\wedge$ and $\vee$) that equation (9.1.1) holds.

3. For an arbitrary game v, split the game into its non-negative part $v \vee \underline{0}$ and its non-positive part $v \wedge \underline{0}$. The transfer property and parts 1 and 2 imply

$$\begin{aligned}\psi(v) &= \psi(v) + \psi(\underline{0}) \\ &= \psi(v \vee \underline{0}) + \psi(v \wedge \underline{0}) \\ &= \sum_{S \subseteq N} [\psi((v \vee \underline{0})(S)\delta_S) + \psi((v \wedge \underline{0})(S)\delta_S)] \\ &= \sum_{S \subseteq N} \psi(v(S)\delta_S).\end{aligned}$$

Hence equation (9.1.1) holds for all TU-games. □

Remark. The converse is also true : If a solution concept ψ on the class G^N of TU-games satisfies equation (9.1.1) for all games $v \in G^N$ then ψ satisfies the transfer property and $\psi(\underline{0}) = 0$.

While lemma 9.1.3 shows that a solution concept satisfying the transfer property is determined by its values on multiples of Dirac games, the next lemma shows it is also determined by its values on multiples of unanimity games.

Lemma 9.1.4 Let N be fixed. Suppose for each $S \in 2^N \setminus \{\emptyset\}$ and for each real number α, a vector $\psi_{\alpha,S} \in \mathbf{R}^N$ is given, satisfying $\psi_{0,S} = 0$ for all $S \in 2^N \setminus \{\emptyset\}$. Then there exists a unique solution concept on G^N satisfying the transfer property, such that

$$\psi(\alpha u_S) = \psi_{\alpha,S} \qquad \text{for all } \alpha \in \mathbb{R}, \text{ and all } S \in 2^N \setminus \{\emptyset\}. \qquad (9.1.2)$$

Proof : First we prove unicity. Suppose there exists a solution ψ satisfying equation (9.1.2) and the transfer property. Then $\psi(\underline{0}) = \psi(0u_N) = \psi_{0,N} = 0$. Hence according to lemma 9.1.3, equation (9.1.1) holds, and applying it to the game αu_S, we obtain

$$\psi_{\alpha,S} = \psi(\alpha u_S) = \sum_{T:T\supseteq S} \psi(\alpha\delta_T) \qquad \text{for all } \alpha \in \mathbb{R}, \text{ for all } S \in 2^N \setminus \{\emptyset\}. \qquad (9.1.3)$$

For each fixed α this finite system of linear equations (with variables $\psi_{\alpha,S}$ and $\psi(\alpha\delta_S)$, $S \in 2^N \setminus \{\emptyset\}$) is easily inversed, yielding

$$\psi(\alpha\delta_T) = \sum_{S:S\supseteq T} (-1)^{|S\setminus T|}\, \psi_{\alpha,S} \qquad \text{for all } \alpha \in \mathbf{R} \text{ and all } T \in 2^N \setminus \{\emptyset\}. \qquad (9.1.4)$$

Hence by equation (9.1.1),

$$\psi(v) = \sum_{T\subseteq N} \sum_{S:S\supseteq T} (-1)^{|S\setminus T|}\, \psi_{v(T),S} \qquad \text{for all TU-games } v, \qquad (9.1.5)$$

which implies ψ is unique.

This construction of ψ proves existence as well : given the numbers $\psi_{\alpha,S}$ for all $\alpha \in \mathbf{R}$ and $S \in 2^N \setminus \{\emptyset\}$, construct a solution ψ first on Dirac games, using equation (9.1.4) and then on all TU-games using equation (9.1.1). This solution ψ will then satisfy equation (9.1.1), hence it satisfies the transfer axiom. It also satisfies equation (9.1.2), so it is the solution concept asked for. □

Using this lemma, we now prove

Theorem 9.1.5 The Shapley value is the unique value on the class G^N of TU-games satisfying efficiency, anonymity, the null player property and the transfer property.

Proof : We already noted that the Shapley value satisfies the four properties. To prove uniqueness, let ψ be a value that satisfies the four properties mentioned. Consider a game of the form αu_S. By the null player property, $\psi_i(\alpha u_S) = 0$ if i is not a member of S, and by anonymity, all players in S obtain the same payoff. Hence,

$$\psi_i(\alpha u_S) = \begin{cases} 0 & \text{if } i \notin S \\ x & \text{if } i \in S \end{cases}$$

for some real number x. Efficiency then yields $|S|x = \alpha u_S(N) = \alpha$ and $x = \alpha/|S|$. Hence ψ is determined on multiples of unanimity games, $\psi(0u_S) = 0$ for all non-empty coalitions S, and lemma 9.1.4 implies uniqueness. □

Another solution concept is the *Banzhaf* value η (cf. Banzhaf (1965), Owen (1975a)), defined on G^N by

$$\eta_i(v) = \sum_{S:i\in S} [v(S) - v(S \setminus \{i\})]$$

for all $i \in N$. It is easily seen that the Banzhaf value satisfies anonymity, additivity and the null player property. Being additive, it satisfies the transfer property as well. Note that it does not satisfy efficiency. Define $\bar{\eta}(v) := \sum_{i \in N} \eta_i(v)$. Now the characterization by Dubey and Shapley (1979) of the Banzhaf value on the class of monotonic simple games can be extended to characterizations on the class of all simple games and the class of all TU-games. Along similar lines as theorems 9.1.1, 9.1.2 and 9.1.5 one can show

Theorem 9.1.6

1. The Banzhaf value is the unique value ψ on the class CG^N of control games satisfying anonymity, the null player property and the transfer property such that

$$\sum_{i \in N} \psi_i(v) = \bar{\eta}(v) \qquad \text{for all } v. \tag{9.1.6}$$

2. The Banzhaf value is the unique value ψ on the class SG^N of simple games satisfying anonymity, the null player property, the transfer property and equation (9.1.6).

3. The Banzhaf value is the unique value ψ on the class G^N of TU-games satisfying anonymity, the null player property, the transfer property and equation (9.1.6).

Other authors have tried to remove the additivity axiom from the axiomatic characterization of the Shapley value; a well-known work in this respect is Young (1985) on monotonic solutions of cooperative games. The precise relation between his notion of strong monotonicity and Dubeys transfer axiom is a possible object of further study.

A different view of power indices is taken in Quint (1993), where powerlessness is measured instead of power. Quint axiomatically characterizes his measures on the class of monotonic simple games, and perhaps this can be generalized to the class of all TU-games.

Chapter 10

Veto-rich TU-games

The aim of this chapter, which is based on Arin and Feltkamp (1994), is the study of veto-rich TU-games. A TU-game (N, v) is a *veto-rich game* if it has at least one veto player. Recall a veto player[1] is a player i such that $i \notin S$ implies $v(S) = 0$.

Clearly, a game (N, v) with veto player i has imputations if and only if $v(N) \geq v(\{i\})$. As in this chapter we are interested in the nucleolus, which is an imputation, we will assume all games have imputations.

The class VG_i^N of veto-rich games with fixed player set N and a fixed veto player i which have imputations is a convex cone in the class of all TU-games, that is, if v and w are (veto-rich) games with veto player i, then so is $\alpha v + \beta w$ for all non-negative numbers α and β.

Subclasses of the class of veto-rich TU-games have been studied by different authors : big boss games by Muto, Nakayama, Potters, and Tijs (1988) and clan games by Potters, Poos, Tijs, and Muto (1989). In these papers several economic illustrations are presented. One important difference between these classes and the class of veto-rich games is that veto-rich games do not have to be monotonic, which permits to model more economic situations.

Other economic illustrations of veto agents are the following. A market with increasing returns to scale, where the agents are one monopolist and $n - 1$ consumers has been studied by Sorenson, Tschirhart, and Whinston (1978). An information market with one possessor of information and many demanders has been studied by Muto, Potters, and Tijs (1989). A variant of this market, where demanders compete, which destroys the monotonicity of the games of Muto, Potters, and Tijs (1989), is considered in Arin (1992). Different types of auctions have been modeled as a veto-rich game, see Schotter (1974) and Graham, Marshall, and Richard (1990). Also, production economies with a landowner and landless peasants (cf. Shapley and Shubik (1967)) can be modeled

[1]Note that we do not require a veto-rich game to be non-negative. However, the name 'veto player' seems to imply a veto player occupies a strong position, which is not true if some coalitions containing the veto players have negative worth while all coalitions which do not contain all veto players have zero worth.

as games with a veto player. Chetty, Dasgupta, and Raghavan (1976) computed the nucleolus of these games.

In the present paper we exploit the special properties of veto-rich games to compute the nucleolus, introduced by Schmeidler (1969) and the kernel, introduced by Davis and Maschler (1965). The nucleolus was introduced as the unique imputation that lexicographically minimizes the vector of non-increasingly ordered excesses over the set of imputations. Peleg [see Kopelowitz (1967)] suggested a "translation" of the definition of the nucleolus into a sequence of linear programs. Theoretically, this sequence could have length 2^n, but usually, it terminates long before that. Kohlberg (1972) developed another method to locate the nucleolus. His approach involves a single, but extremely large linear program ($O(n)$ variables and $2^n!$ constraints for a n-person game). Moreover, the coefficients appearing in the constraints have a very wide range, causing serious numerical difficulties even for four players. In Owen's (1974) improved version one has to solve a single linear program with $O(2^n)$ variables and 4^n constraints. Maschler, Peleg, and Shapley (1979) gave a constructive definition of the nucleolus, in which the set of imputations under consideration is iteratively reduced until only one imputation remains. This approach leads to $O(4^n)$ linear programs, each with $O(n)$ variables and $O(2^n)$ constraints including only coefficients of $-1, 0$ or 1. Sankaran (1991) proposed a similar procedure, with only $O(2^n)$ iterations. These formulations are numerically more stable than the approach of Kohlberg and Owen, but the number of linear programs is enormous.

On special classes of games, it may be possible to take advantage of the specific structure of the games to compute the nucleolus using a more efficient algorithm. For example, Solymosi and Raghavan (1994) propose an algorithm for computing the nucleolus of assignment games. In these games, there are two types of players. If there are m players of the first type, n players of the second type, and $m = \min\{m, n\}$, then Solymosi and Raghavan's algorithm computes the nucleolus in at most $m(m+3)/2$ steps, each requiring at most $O(m.n)$ elementary operations. They apply graph-related techniques instead of linear programming.

Granot and Huberman (1984) proved that for minimum cost spanning tree games the size of the linear programs in the algorithm of Maschler, Peleg, and Shapley can be reduced : the coalitions whose complement is not connected in the tree constructed for the grand coalition are not relevant for the computation of the nucleolus. Moreover they provide a geometric characterization of the nucleolus, which they exploit to give a sequence of vectors that converges to the nucleolus.

Granot, Maschler, Owen, and Zhu (1994) study the kernel and nucleolus of standard tree games. These games are convex, so the kernel and nucleolus coincide. They give an algorithm that gives the nucleolus in n steps in a standard tree game with n players.

Rosenmüller and Sudhölter (1994) compute the nucleolus of homogeneous games with steps.

Huberman (1980) proved that the nucleolus of an arbitrary game only depends on so-called essential coalitions if the core is non-empty. In minimum cost spanning tree games, these are exactly the coalitions that are used by Granot and Huberman. Derks and Kuipers (1992) use this to find an $O(nc^2)$ algorithm for computing the nucleolus of a game with a particular connectedness property that has a non-empty core. Here, c is the number of connected coalitions. A veto-rich game can be viewed as having 2^{n-1} connected coalitions, so their algorithm is $O(n4^{n-1})$ on the class of veto-rich games.

An extensive overview of the research on the nucleolus is given in Maschler (1992).

Recently, Potters, Reijnierse, and Ansing (1994) presented a a 'prolonged' simplex algorithm to compute the nucleolus, that is very fast. For a five person example of Sankaran (1991), it uses only 11 pivot operations, about 2100 elementary steps, and 0.07 seconds CPU time on a SPARC/SUN/10/41 station.

This chapter is organized as follows : we introduce the nucleolus and the kernel of a TU-game in section 10.1. Section 10.2 contains the proof that the kernel of a veto-rich game only contains the nucleolus. In section 10.3 we use this result to present an algorithm that computes the nucleolus of an n-player veto-rich game in at most n stages, each stage requiring taking the minimum of not more than 2^{n-1} real numbers obtained by n additions, 2 subtractions and one division. In each stage at least one coordinate of the nucleolus is computed. Section 10.4 concludes with a short study of other solution concepts on veto-rich games : we show that for arbitrary veto-rich games the nucleolus is not a population monotonic allocation scheme in the sense of Sprumont (1990), nor do the Shapley value, τ-value or nucleolus coincide. As shown in Potters, Muto, and Tijs (1990), the bargaining set and the core of a veto-rich game coincide if the core is non-empty.

10.1 Basic definitions

Let (N,v) be a TU-game. For an imputation $x \in I(N,v)$, define the *excess* of a coalition $S \subseteq N$ at x as $E(S,x) = v(S) - x(S)$ and let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order of magnitude. The lexicographic order $\prec_L$ between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$, and the weak lexicographic order $\preceq_L$ by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a TU-game as the unique imputation that lexicographically minimizes the vector of non-increasingly ordered excesses over the set of imputations $I(N,v)$. In formula :

$$\{\nu(N,v)\} = \{x \in I(N,v) \mid \theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N,v)\}.$$

It is well known that the nucleolus $\nu(N,v)$ lies in the core of the game (N,v), provided that this core is nonempty.

For two players i,j of a TU-game (N,v) and an allocation x, define the complaint of

i against j at allocation x by

$$s_{ij}(x) = \max\{E(S,x) \mid i \in S \not\ni j\}.$$

It is the maximal excess of a coalition that contains i but not j. The idea captured by the kernel is that if at an imputation x, the complaint of a player against any other player is less than the complaint of this other player against the first player, then the first player should get less. Of course, the players cannot get less than their individual worths if x is an imputation, so the kernel is defined as

$$\mathcal{K}(N,v) = \{x \in I(N,v) \mid \forall i,j \in N : s_{ij}(x) \geq s_{ji}(x) \text{ or } x_i = v(\{i\})\}.$$

The kernel of a game (N,v) always contains the nucleolus $\nu(N,v)$.

We denote the complement $N \setminus S$ of a coalition S by S^c.

10.2 The kernel

In this section, we concentrate on the kernel of a veto-rich game and prove it consists of only one imputation, which then has to be the nucleolus. The proof is based on the crucial fact that if i is a veto player and j another player in a veto-rich game (N,v), then $E(S,x) = -\sum_{k\in S} x_k \leq -x_j = E(\{j\},x)$ for all imputations x and all coalitions S containing player j but not the veto player i. Hence $s_{ji}(x) = -x_j$.

Lemma 10.2.1 Let x lie in the kernel of the veto-rich game (N,v). Then

$$x_i - v(\{i\}) \geq x_j$$

for any veto player i and any player j.

Proof : Suppose an inputation x satisfies $x_i - v(\{i\}) < x_j$. Then $s_{ij}(x) \geq v(\{i\}) - x_i > -x_j = s_{ji}(x)$ and $x_j > x_i - v(\{i\}) \geq 0$, because x is an imputation. So x does not lie in the kernel. □

Recall a game (N,v) is essential if $v(N) \neq \sum_{i\in N} v(\{i\})$. Note that in an essential veto-rich game, any veto player i is allocated strictly more than his individual worth $v(\{i\})$ in a kernel element x. This is easily seen : by lemma 10.2.1, $x_i - v(\{i\})$ is larger than or equal to x_j for any other player j and if i gets a payoff of $v(\{i\})$, then all other players get 0. But then $v(N) = x(N) = v(\{i\}) = \sum_{j\in N} v(\{j\})$, so the game is inessential. Hence, it holds that $s_{ij}(x) \geq s_{ji}(x)$ for all other players j.

Second, if $v(\{i\}) > 0$ in a veto-rich game (N,v) with veto player i, then this veto player gets strictly more than any other player in a kernel element.

Third, if there are two or more veto players, their payoffs are equal in a kernel element. Obviously, in this case, the individual worths of the veto players are zero. It can also happen that though there is only one veto player, there is another player who gets the same payoff as the veto player, as is shown by the following example :

Example 10.2.2 Let $N = \{0,1,2\}$, let 0 be a veto player, and let $v(\{0\}) = 0$, $v(\{0,1\}) = 1 = v(\{0,2\})$, $v(N) = 3$. Then the unique kernel element is the equal split $(1,1,1)$.

The next lemma determines the unique kernel payoff of a certain kind of players.

Lemma 10.2.3 If x lies in the kernel of the veto-rich game (N, v) and $v(S) \geq v(N)$ for a coalition S containing a veto player i, then $E(S,x) \geq 0$ and $x_j = 0$ for all players j in the complement of S.

Proof : Let j lie in the complement of S. Because S contains a veto player i,

$$s_{ij}(x) \geq E(S,x) = v(S) - x(S) \geq v(N) - x(S). \tag{10.2.1}$$

Because $x_k \geq v(\{k\}) = 0$ for $k \neq i$, it follows that $v(N) - x(S) \geq v(N) - x(N) = 0$. Combining this with equation 10.2.1, we obtain

$$s_{ij}(x) \geq 0 \geq -x_j = s_{ji}(x). \tag{10.2.2}$$

If x lies in the kernel, either inequality 10.2.2 is an equality, or $x_j = v(\{j\}) = 0$. But if inequality 10.2.2 is an equality, then $x_j = 0$ as well. □

Lemma 10.2.4 If x lies in the kernel of the veto-rich game (N, v) with veto player i, and $v(S) < v(N)$ for a coalition S containing veto player i, then $E(S,x) < 0$.

Proof : Suppose that $E(S,x) \geq 0$. For any $j \in N \setminus S$, coalition S can be used by the veto player to complain against j. Now $s_{ji}(x) = -x_j \leq 0$, so either $x_j = v(\{j\}) = 0$, or $0 \geq s_{ji}(x) \geq s_{ij}(x) \geq 0$, in which case $s_{ji}(x) = 0$. But then $x_j = 0$ as well. So all players outside S are allocated 0. Then the excess of S equals $v(S) - x(S) = v(S) - v(N) + x(N \setminus S) = v(S) - v(N) < 0$. This is a contradiction. □

The next corollary asserts that in an essential veto-rich game with veto player i the players other than veto player i whose payoffs were not determined in lemma 10.2.3 get positive payoffs in any kernel element.

Corollary 10.2.5 If x lies in the kernel of the game (N, v) with veto player i, and if for another player j, there is no coalition $S \subseteq N \setminus \{j\}$ with $i \in S$ and $v(S) \geq v(N)$, then $x_j > 0$.

Proof : By lemma 10.2.1, if $x_i = v(\{i\})$ for a kernel element x, then the game has to be inessential, so $v(\{i\}) = v(N)$, contradicting the hypothesis. Hence, $x_i > v(\{i\})$, which implies $s_{ij}(x) \geq s_{ji}(x) = -x_j$. Now $s_{ij}(x) = E(S,x)$ for some coalition S containing player i but not player j. By assumption, $v(S) < v(N)$, so by lemma 10.2.4, the excess of S is strictly negative and hence $x_j > 0$. □

The importance of this result lies in the fact that for a player j that has a positive payoff in a kernel element, the complaint of j against a veto player i has to equal the complaint of i against j. So the inequalities in the definition of the kernel can be replaced by equalities, which makes the process of determining the kernel easier.

Before we give the main theorem of this section, we compute the kernel of a veto-rich game that arises from an auction with an auctioneer who sells an indivisible object in an auction with many bidders.

Example 10.2.6 Let $N = \{0, \ldots, n\}$ and let the auctioneer (player 0) valuate the object at $a_0 = 0$, while this value is $a_j \geq 0$ to the other players $j \in N$. The worth $v(S)$ of a coalition S is zero if this coalition does not contain the auctioneer, and $v(S) = \max\{a_j \mid j \in S\}$ otherwise.

Let a player with the highest valuation be called h and let a player with the highest remaining valuation after h has been eliminated be called s. Suppose $a_h \geq a_s \geq 0$ and $a_h > 0$. Now $v(\{0, h\}) = v(N)$, so lemma 10.2.3 implies that a kernel element x has to satisfy $x_j = 0$ if $j \notin \{0, h\}$. If $a_s = a_h$, then also $x_h = 0$, and the seller gets all, i.e. $x_0 = a_h$. On the other hand, if $a_s < a_h$, then there is no coalition S not containing player h with $v(S) \geq v(N)$, so by corollary 10.2.5, $x_h > 0$. Remembering the remark after the corollary, we obtain $-x_h = s_{h0}(x) = s_{0h}(x)$. Any coalition S containing the auctioneer but not player h has excess $E(S, x) = v(S) - x_0$, which is highest if player s is an element of S. Hence $s_{0h}(x) = E(\{0, s\}, x) = a_s - x_0$, which implies

$$x_h = x_0 - a_s.$$

Together with efficiency ($x_0 + x_h = v(N) = a_h$), this implies $x_0 = (a_h + a_s)/2$ and $x_h = (a_h - a_s)/2$.

So according to the kernel, the object is sold to the bidder with highest valuation and the price is the average of the highest and second highest valuation.

In the example, the kernel is a singleton. That this is not a coincidence is shown in the following theorem.

Theorem 10.2.7 The kernel of a veto-rich game consists of a unique element.

Proof : Let x be a kernel element of the veto-rich game (N, v) with veto player i. By lemma 10.2.3, we know that $x_j = 0$ for players j other than veto player i such that there exists a coalition $S \subseteq N \setminus \{j\}$ with $i \in S$ and $v(S) \geq v(N)$. Denote the set of players whose payoffs are determined in this way by D_0.

Suppose that there are still players other than veto player i whose payoffs have not yet been determined (if not, go to the last paragraph of this proof). Then from the remark after corollary 10.2.5, we know $s_{ij}(x) = s_{ji}(x) = -x_j$ for all players $j \neq i$ whose payoffs are not yet determined. We now iteratively, in at most $|N|$ stages, determine more and more coordinates of x, until all coordinates are determined. As x was chosen arbitrarily in the kernel, this proves that the kernel contains only one element, x.

Consider a stage $t \geq 1$. Let the set D_{t-1} consist of the players whose payoffs have been uniquely determined before stage t. If there are still players other than the veto player i whose payoffs remain to be determined, consider the set of coalitions admissible at stage t

$$\mathcal{A}_t = \{S \subseteq N \mid i \in S \text{ and there exists a player } j \in D_{t-1}^c \setminus S\} \tag{10.2.3}$$

and the subset of coalitions with maximal excess

$$\mathcal{M}_t := \operatorname{argmax}\{E(S,x) \mid S \text{ is admissible at stage } t\} \tag{10.2.4}$$

and the coalition

$$S_t = \bigcap\{S \mid S \in \mathcal{M}_t\}.$$

Furthermore, denote $p_t := -E(S,x)$ for an $S \in \mathcal{M}_t$.

By construction, for a player $j \in D_{t-1}^c$, there exists no coalition containing player i but not player j with excess higher than $-p_t$. Furthermore, if $j \in D_{t-1}^c \setminus S_t$, there exists a coalition $S \in \mathcal{M}_t$ not containing player j. Hence, $E(S,x) = s_{ij}(x) = -p_t$. Vice versa, for a coalition $S \in \mathcal{M}_t$, there exists a player in S^c that has not yet been allocated, and for any such player j, there exists no coalition containing player i but not player j, with excess higher than $E(S,x)$, so $s_{ij}(x) = E(S,x)$. Hence by using the coalitions $S \in \mathcal{M}_t$, we can exactly determine the complaints of player i against the players $j \in D_{t-1}^c \setminus S_t$.

Now take $S \in \mathcal{M}_t$. Then $-x_j = s_{ji}(x) = s_{ij}(x) = E(S,x) = -p_t$ for any $j \in D_{t-1}^c \setminus S$. So all players outside S_t whose payoffs were not yet determined have the same payoff p_t.

We still have to prove that this payoff p_t is independent of the allocation x. Now for $S \in \mathcal{M}_t$,

$$\begin{aligned}
-p_t &= E(S,x) \\
&= v(S) - x(S) \\
&= v(S) - v(N) + x(N \setminus S) \\
&= v(S) - v(N) + x(D_{t-1} \setminus S) + x(D_{t-1}^c \setminus S) \\
&\overset{*}{=} v(S) - v(N) + x(D_{t-1} \setminus S) + |D_{t-1}^c \setminus S| \cdot p_t,
\end{aligned}$$

where $\overset{*}{=}$ follows because all players in $D_{t-1}^c \setminus S$ are allocated p_t. Hence,

$$p_t = \frac{v(N) - v(S) - x(D_{t-1} \setminus S)}{|D_{t-1}^c \setminus S| + 1} = q_t(S) \tag{10.2.5}$$

for all $S \in \mathcal{M}_t$, where $q_t(T)$ is defined by

$$q_t(T) := \frac{v(N) - v(T) - x(D_{t-1} \setminus T)}{|D_{t-1}^c \setminus T| + 1} \tag{10.2.6}$$

for all $T \in \mathcal{A}_t$. We will prove that $p_t = \min\{q_t(T) \mid T \in \mathcal{A}_t\}$, which is independent of the choice of kernel element x, because x_i was uniquely determined by the previous stages for $i \in D_{t-1}$ and the set $\mathcal{A}_t$ is likewise determined by D_{t-1}.

For $j \in D_{t-1}^c$ it holds that $-x_j = s_{ji}(x) = s_{ij}(x) \leq -p_t$ by definition of p_t. Hence, $x_j \geq p_t$ for all $j \in D_{t-1}^c$. So for T admissible

$$\begin{aligned} -p_t &\geq E(T,x) \\ &= v(T) - x(T) \\ &= v(T) - v(N) + x(D_{t-1} \setminus T) + x(D_{t-1}^c \setminus T) \\ &\geq v(T) - v(N) + x(D_{t-1} \setminus T) + p_t |D_{t-1}^c \setminus T|, \end{aligned}$$

which implies that $p_t \leq q_t(T)$ and hence that $p_t = \min\{q_t(T) \mid T \in \mathcal{A}_t\}$.

Note that for $S \in \mathcal{M}_t$ it holds that $p_t = -E(S,x) > p_{t-1}$, because if S is admissible at stage t then S was admissible at stage $t-1$ but did not have maximal excess.

When the payoffs of all players other than veto player i have been uniquely determined efficiency implies $x_i = v(N) - x(N \setminus \{i\})$, so the payoff of player i is then also uniquely determined. □

Corollary 10.2.8 Let (N,v) be a veto-rich game. Then $\mathcal{K}(N,v) = \{\nu(N,v)\}$.

Proof : The nucleolus lies in the kernel, which consists of a unique element. □

It has to be noted that although we have singled out a veto player in the proof of theorem 10.2.7, the kernel is independent of *which* veto player has been singled out.

10.3 The nucleolus

The proof of theorem 10.2.7 gives insight in the structure of the kernel/nucleolus, and suggests an algorithm to compute the nucleolus of a veto-rich game with a veto player i.

The idea is as follows : begin by assigning zero to those players j such that there is a coalition S containing i but not j, that satisfies $v(S) \geq v(N)$. Call the set of these players A_0.

Then iteratively, at each step t, look for the coalitions S containing i, that still have players in their complement whose payoffs have not yet been assigned. Among these *admissible* coalitions, select those coalitions that minimize the amount

$$\frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A_{t-1}^c| + 1}.$$

The idea is that for any such minimizing coalition S, the amount $v(N) - v(S) - x(A_{t-1} \setminus S)$ remains to be divided, and dividing it equally between the not yet allocated players outside S and the coalition S itself, will equate the complaints of the veto player i against the players outside S that have just been allocated and the complaints of those players against player i. Let A_t equal the set of players whose payoffs have been determined in step t or earlier.

When all other players have been assigned payoffs in this way, the veto player i obtains the rest. We now give a more formal description.

Algorithm 10.3.1 (Nucleolus for veto-rich games)

input : a veto-rich game (N, v) with a veto player i
output : an allocation x (the nucleolus of the game)

0. Start with the stage $t = 0$. Define the set of people whose payoff is allocated in stage 0 :

   $$A_0 := \{j \in N \setminus \{i\} \mid \exists S \subseteq N \setminus \{j\} : i \in S \text{ and } v(S) \geq v(N)\}.$$

   Put $q_0 = 0$ and allocate $x_j = q_0 = 0$ for all $j \in A_0$.

1. While there is a player that is not the veto player i and whose payoff has not been allocated, do steps 1a to 1d

   (a) Put $t := t + 1$.

   (b) Given the set A_{t-1} of players whose payoffs have been allocated before stage t, call a coalition S *admissible at stage* t if S contains the veto player i and there remain players in $N \setminus S$ to be allocated. For all admissible coalitions S, recall

   $$q_t(S) = \frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A_{t-1}^c \setminus S| + 1}.$$

   (c) Define the payoff obtained by players whose payoff is allocated at stage t

   $$q_t := \min\{q_t(S) \mid S \text{ admissible at stage } t\},$$

   the set of players who are not going to be allocated during this stage

   $$S_t := \bigcap \operatorname{argmin}\{q_t(S) \mid S \text{ admissible at stage } t\}$$

   and the set of players allocated at or before stage t

   $$A_t := A_{t-1} \cup S_t^c = A_{t-1} \cup (A_{t-1}^c \setminus S_t).$$

   (d) Allocate $x_j = q_t$ for all $j \in A_t \setminus A_{t-1} = A_{t-1}^c \setminus S_t$.

2. Allocate $x_i = v(N) - x(N \setminus \{i\})$ to veto player i.

3. Define $x = x(N, v)$ as the vector with coordinates $(x_j)_{j \in N}$.

In each stage (except maybe in stage 0), at least one player is allocated, so at the latest after stage $|N|$, each player has been allocated a payoff. Before we prove that the algorithm yields the nucleolus, we need a lemma.

Lemma 10.3.2 If the algorithm allocated a payoff to player k before player j, then $x_k \leq x_j$, unless[2] $j = i$ and $v(\{i\}) < 0$.

Proof : Let (N, v) be a veto-rich game with veto player i. If j (and hence k) do not coincide with player i, then it is sufficient to prove $q_t > q_{t-1}$ for all stages $t > 0$. Let $t = 1$ and let S be a coalition. If $v(S) \geq v(N)$, then all players outside S were allocated 0 in stage 0, so coalition S is not admissible at stage 1. So any coalition S admissible at stage 1 satisfies $v(S) < v(N)$, which implies

$$q_1(S) = \frac{v(N) - v(S)}{|A_0^c \setminus S| + 1} > 0.$$

Hence

$$q_1 = \min_{S \in \mathcal{A}_1} q_1(S) \geq \min_{S: v(S) - v(N) < 0} q_1(S) > 0 = q_0.$$

Let $t > 1$ and suppose there remain players to be allocated at stage t. Let S be an admissible coalition. Then at stage $t-1$, coalition S was admissible too, but was not used to determine q_{t-1}, so

$$q_{t-1} < q_{t-1}(S) = \frac{v(N) - v(S) - x(A_{t-2} \setminus S)}{|A_{t-2}^c \setminus S| + 1},$$

hence

$$\begin{aligned} v(N) - v(S) - x(A_{t-2} \setminus S) &> (|A_{t-2}^c \setminus S| + 1) \cdot q_{t-1} \\ &= (|A_{t-1}^c \setminus S| + |(A_{t-1} \setminus A_{t-2}) \setminus S| + 1) \cdot q_{t-1}. \end{aligned}$$

Now $x_k = q_{t-1}$ for $k \in (A_{t-1} \setminus A_{t-2}) \setminus S$, so transferring $(|(A_{t-1} \setminus A_{t-2}) \setminus S|) \cdot q_{t-1} = x((A_{t-1} \setminus A_{t-2}) \setminus S)$ to the left-hand side, we obtain

$$\begin{aligned} v(N) - v(S) - x(A_{t-1} \setminus S) &= v(N) - v(S) - x(A_{t-2} \setminus S) - x((A_{t-1} \setminus A_{t-2}) \setminus S) \\ &> (|A_{t-1}^c \setminus S| + 1) q_{t-1}, \end{aligned}$$

which implies that

$$q_t(S) = \frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A_{t-1}^c \setminus S| + 1} > q_{t-1}.$$

Hence also $q_t > q_{t-1}$, as q_t is the minimum of $q_t(S)$ over all admissible coalitions S.

Finally, we prove that veto player i has a payoff larger than or equal to that of the other players if $v(\{i\}) \geq 0$. If all other players are allocated payoffs at stage 0, then they all get the same payoff zero. Hence, $x_i = v(N)$, which because we assumed that there are imputations, has to be at least equal to $v(\{i\})$, which is assumed to be non-negative. Hence i's payoff is larger than or equal to that of the other players. If not all other

[2] This lemma will only be needed for $j \neq i$.

players are allocated zero, then in the stage t where the payoff of the last player j other than i is allocated, coalition $\{i\}$ is admissible. Then

$$x_j = q_t \leq q_t(\{i\}) = \frac{v(N) - v(\{i\}) - x(A_{t-1})}{|A_{t-1}^c \setminus \{i\}| + 1} \leq \frac{v(N) - x(A_{t-1})}{|A_{t-1}^c|}. \tag{10.3.1}$$

The last fraction is what the not yet allocated players (including veto player i) would get if $v(N) - x(A_{t-1})$ were divided equally. The not yet allocated players other than i get at most this amount from the algorithm, so player i must get at least this amount from the algorithm, hence $x_i \geq x_j$. By the first part of this proof, j gets as least as much as the other players (except i), so i gets more than any other player. Together with the first part of the proof, this proves the lemma. □

Note that if there are two or more veto players in a game, the veto players that are not singled out by the algorithm get the same payoff as the veto player i that is singled out, so their payoff is allocated in the last iteration of step 1 : any players allocated at a later iteration would have to get strictly more by the proof of lemma 10.3.2, which is impossible by lemma 10.2.1.

Theorem 10.3.3 The allocation x defined in the algorithm is the nucleolus.

Proof : This theorem can be proved directly using Kohlberg's (1971) characterization of the nucleolus, but we prove the theorem by proving that the allocation is the unique kernel element. Let i be a veto player of the game (N, v) and apply the algorithm to (N, v), with i as the special veto player.

First, the algorithm allocates a zero payoff to any player $j \neq i$ such that there exists a coalition S which contains i but not j and that satisfies $v(S) \geq v(N)$. So the set of players A_0 that are allocated a payoff of zero in the first stage of the algorithm, coincides with the set of people D_0 whose payoff is determined to be zero in the first step of theorem 10.2.7.

Suppose that up to stage $t-1$, exactly those players have been allocated whose payoffs are determined in theorem 10.2.7 and that these players have exactly been allocated their kernel payoffs. Then a coalition is admissible in stage t of the algorithm if and only if it is admissible in the same stage of theorem 10.2.7.

Because $A_{t-1} = D_{t-1}$, equation 10.2.5 implies that $p_t = q_t(S) = q_t$ for all coalitions $S \in \mathcal{M}_t$. It remains to be proved that if T is admissible at stage t and $E(T, \nu) < -p_t = \max\{E(U, \nu) \mid U \text{ admissible at stage } t\}$, then $q_t(T) > p_t$. For this, take a coalition T admissible at stage t such that $E(T, \nu) < -p_t$. Rewriting the excess of T, we obtain $-p_t > v(T) - v(N) + x(A_{t-1} \setminus T) + x(A_{t-1}^c \setminus T)$. By lemma 10.3.2, we know that all players $j \neq i$ who have not yet been allocated will be allocated payoffs that are larger than or equal to $q_t = p_t$ by the algorithm. Hence because $i \in T$, $-p_t > v(T) - v(N) + x(A_{t-1} \setminus T) + p_t \cdot |A_{t-1}^c \setminus T|$, which implies

$$p_t < \frac{v(T) - v(N) + x(A_{t-1} \setminus T)}{|A_{t-1}^c \setminus T| + 1} = q_t(T).$$

So $q_t(T)$ attains its minimum $q_t = p_t$ only at those admissible coalitions that have maximal excess amongst the admissible coalitions. But then exactly those players whose payoffs were determined in this stage in theorem 10.2.7 will be allocated in this stage of the algorithm and furthermore, they are allocated their kernel payoff p_t, which is positive.

So the players other than the veto player i are allocated their kernel payoffs. And in step 2, player i is allocated the remainder, which is exactly player i's kernel payoff. □

Example 10.3.4 Consider the five person game shown in the first two rows of table 10.1, in which player 0 is a veto player. We compute its nucleolus in the three last rows. The

S	0	0,1	0,2	0,3	0,4	0,1,2	0,1,3	0,1,4	0,2,3
$v(S)$	0	3	2	1	1	5	5	4	4
$q_1(S)$	2	7/4	2	9/4	9/4	5/3	5/3	2	2
$q_2(S)$	9/4	**2**	7/3	8/3	9/4	**2**	**2**	**2**	5/2
$q_3(S)$	**5/2**	—	**5/2**	3	**5/2**	—	—	—	**5/2**

S	0,2,4	0,3,4	0,1,2,3	0,1,2,4	0,1,3,4	0,2,3,4	N
$v(S)$	3	2	8	0	0	0	10
$q_1(S)$	7/3	8/3	**1**	5	5	5	—
$q_2(S)$	7/3	8/3	—	5	5	5	—
$q_3(S)$	**5/2**	3	—	—	—	5	—

Table 10.1: The algorithm applied to a five person game.

minimum in each row is printed in boldface. In the first stage, the minimum $q_1 = 1$ is attained at coalition $\{0,1,2,3\}$, hence player 4 (the unique player in the complement) is assigned $x_4 = 1$. At stage 2, only coalitions which do not contain $\{1,2,3\}$ are taken into consideration. The minimum $q_2 = 2$ is attained at coalitions $\{0,1\}$, $\{0,1,2\}$, and $\{0,1,3\}$, so all players outside the intersection of these coalitions, i.e. players 2 and 3, are assigned $x_2 = x_3 = 2$. In stage 3 only the coalitions which do not contain player 1 are taken into account to compute $x_1 = q_3 = 2.5$. Finally, the veto player 0 gets the rest, so $x_0 = 10 - 1 - 2 - 2 - 2.5 = 2.5$. Hence, the nucleolus of this game equals $(2.5, 2.5, 2, 2, 1)$.

10.4 Other solution concepts

We now turn our attention to other solution concepts.

Proposition 10.4.1 For a veto-rich game (N, v) with veto player i, the following are equivalent :

1. $\text{Core}(v) \neq \emptyset$.

2. v satisfies $v(S) \leq v(N)$ for all $S \subseteq N$ with $i \in S$.

Proof : As $v(\{j\}) = 0$ for all players $j \in N \setminus \{i\}$, it is clear that '$v(S) \leq v(N)$ for all coalitions S containing the veto player' is a necessary condition for the game to have a non-empty core. That it is also sufficient is shown by the next allocation x : let $x_j = 0$ for $j \neq i$ and let $x_i = v(N)$. Then $x(S) = v(N) \geq v(S)$ if $i \in S$ and $x(S) = 0 = v(S)$ if $i \notin S$. Hence $x \in \text{Core}(v)$. □

Furthermore, if the core of a veto-rich game is not empty, it coincides with the bargaining set $\mathcal{M}_1^i(\{N\})$, as defined in Aumann and Maschler (1964). Maschler's proof of coincidence of core and bargaining set for non-negative veto-rich games (see Potters, Muto, and Tijs (1990)) can easily be extended to the class of all veto-rich games.

Note that our algorithm computes the nucleolus of a veto-rich game even if the core of the game is empty. When the core of a game is non-empty, it is known that the nucleolus coincides with the prenucleolus. It would seem that a slight modification of our algorithm could yield the prekernel and prenucleolus of a non-balanced game, but the obvious modification of eliminating step 0 of the algorithm does not yield the prenucleolus. The problem is that if the core is empty, some players get a negative payoff in the prekernel. Hence, non-veto players will use all players (not equal to i) with negative payoff to complain against a veto player i. This implies that the prekernel is not determinated by the complaints between a veto player and a non-veto player alone, one will also have to know complaints between the players who get negative payoffs. It is not even clear that the set of players who get negative payoffs or even the total payoff to this set is constant in the prekernel, which can contain other allocations than the prenucleolus, as is shown by the next example.

Example 10.4.2 Let $N = \{0,1,2,3,4\}$, let 0 be the veto player, and define the characteristic function v by $v(\{0,1,2\}) = v(\{0,2,3\}) = v(\{0,3,4\}) = v(\{0,4,1\}) = 18$ and $v(N) = 12$. The core of this game[3] is empty and its prekernel contains the set

$$\{(16, -1-\alpha, -1+\alpha, -1-\alpha, -1+\alpha) \mid \alpha \in [-1,1]\}.$$

Indeed, for any allocation x of this form

$$\begin{aligned} E(\{1,2,3,4\},x) \;=\; 4 \;&=\; E(\{0,1,2\},x) \\ &=\; E(\{0,2,3\},x) \\ &=\; E(\{0,3,4\},x) \\ &=\; E(\{0,4,1\},x) \end{aligned}$$

so $s_{ij}(x) = 4$ for all players i and j. The prenucleolus equals $(16,-1,-1,-1,-1)$.

[3]This game was suggested by Gooni Orshan.

We conjecture that once the negative coordinates of the prenucleolus have been determined, the other coordinates can be determined by using algorithm 10.3.1 (leaving out the first step).

For general veto-rich games, the nucleolus does not have to coincide with the τ-value, nor with the Shapley value. This can be seen in the following games.

Example 10.4.3 Let $N = \{0,1\}$, let $v(\{0\}) = 10$, $v(\{1\}) = 0$ and $v(\{0,1\}) = 5$. Here $\nu(N,v) = (5,0)$, the Shapley value is $\phi(N,v) = (7.5,-2.5)$ and the τ-value does not even exist, because the game is not quasi-balanced.

Even if we restrict ourselves to convex veto-rich games the τ-value, Shapley value and nucleolus need not coincide.

Example 10.4.4 Let $N = \{0,1,2\}$, let $v(\{0\}) = 1$, $v(\{0,1\}) = 2 = v(\{0,2\})$, $v(\{0,1,2\}) = 6$ and let the values of the other coalitions equal zero. Then $\nu(N,v) = (8,5,5)/3$, $\tau(N,v) = (38,20,20)/13$ and $\phi(N,v) = (3,1.5,1.5)$.

This in contrast with the result in Muto, Nakayama, Potters, and Tijs (1988) that on the subclass of big boss games, the nucleolus coincides with the τ-value and moreover that if the game is a convex big boss game, then the Shapley value coincides with the nucleolus as well.

Recall that a population monotonic allocation scheme of a game (N,v) is a collection $x = \{x_{jS} \mid j \in S \subseteq N\}$ which satisfies the following two conditions.

- $x_S(S) := \sum_{j\in S} x_{jS} = v(S)$ for all $S \subseteq N$.
- $x_{jS} \leq x_{jT}$ if $j \in S \subseteq T$.

Sprumont (1990) proves that a TU-game (N,v) that has a PMAS x is totally balanced. Proposition 10.4.1 implies that for a game (N,v) with veto player i total balancedness is equivalent to

$$v(T) \leq v(S) \qquad \text{if } i \in T \subseteq S. \tag{10.4.1}$$

Moreover, a game with veto player i satisfying the inequalities 10.4.1 has a PMAS : define $x_{iS} := v(S)$ if S contains i and $x_{jS} := 0$ for all other players j and all coalitions S containing j. Hence, we have the following theorem.

Theorem 10.4.5 The following are equivalent for a game (N,v) with veto player i :

- (N,v) has a PMAS.
- (N,v) is totally balanced.
- (N,v) satisfies the inequalities 10.4.1.

The (extended) nucleolus of a game (N, v) is a PMAS if the set $\{\nu_{jS} \mid j \in S \subseteq N\}$ forms a PMAS, where $\nu_{jS} = \nu_j(S, v_S)$ is the coordinate of player j in the nucleolus of the subgame (S, v_S). The next example shows that there exist veto-rich games which have a PMAS, in which the extended nucleolus is not a PMAS.

Example 10.4.6 Consider the game $(\{0,1,2\}, v)$, defined by $v(\{0,1\}) = v(\{0,2\}) = v(N) = 2$, and $v(S) = 0$ for all other coalitions S. This game is monotonic, so it has a PMAS, but the extended nucleolus is not a PMAS, because it violates the second condition for a PMAS : $\nu_{\{0,1\}} = (1, 1, -)$, $\nu_{\{0,2\}} = (1, -, 1)$, while $\nu_N = (2, 0, 0)$.

Two solutions that are related to the nucleolus and the kernel are the per capita nucleolus and the per capita kernel. They are based on the per capita excesses of coalitions instead of the usual excesses. The per capita excess of a coalition is defined as the quotient of the excess of the coalition and the number of elements of the coalition. Up to now, no algorithm generating the per capita nucleolus has been found.

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Index

Samenvatting

Dit proefschrift, getiteld coöperatie in gecontroleerde netwerkstrukturen, bestudeert met behulp van speltheoretische aanpak samenwerkingsvraagstukken in economische situaties met een netwerkstruktuur, waarvan onderdelen worden gecontroleerd door economische agenten. Het bestaat uit twee delen die onafhankelijk van elkaar gelezen kunnen worden.

Deel I behandelt situaties waarin een groep gebruikers van een voorziening op zo goedkoop mogelijke wijze moet (wil) verbonden worden met de leverancier van die voorziening. Als de kosten van een netwerk dat iedere gebruiker met de leverancier verbindt gedragen moet worden door de gebruikers, is het logisch niet alleen het netwerkconstructieprobleem te beschouwen, maar ook het bijbehorende kostentoewijzingsprobleem. In de literatuur worden de constructie van een netwerk met minimum kosten en het kostentoewijzingsprobleem meestal apart behandeld, maar omdat het twee facetten van één probleem zijn, beschouwen we ze tegelijkertijd. Het kostentoewijzingsprobleem wordt met speltheoretische methoden behandeld, en we concentreren ons vooral op de core van geassociëerde spelen.

Hoofdstuk 2 behandelt minimum kosten opspannende boomproblemen en Bird's boomvektoren, die Bird in 1976 voorstelde als oplossing voor de corresponderende kostentoewijzingsproblemen. Er worden twee nieuwe visies op Bird's boomvektoren geïntroduceerd : eerst worden deze boomvektoren axiomatisch gekarakteriseerd en vervolgens wordt er een niet-coöperatief spel gedefinieerd, waarvan de Nash evenwichten corresponderen met de boomvektoren. Voorts worden de boomvektoren geïntegreerd in het algoritme van Prim (1957) en Dijkstra (1959) dat een minimum kosten opspannende boom construeert : zodra een kant geconstrueerd wordt door het algoritme, worden ook de kosten ervan toegewezen.

Deze aanpak suggereert een methode om toewijzingsregels te definiëren die corresponderen met andere algoritmen voor het construeren van minimum kosten opspannende bomen, zoals de algoritmen van Kruskal (1956) en Borůvka (1926).

In hoofdstuk 3 bewijzen we dat de niet-reduceerbare core nauw verwant is met Kruskal's algoritme. Het blijkt dat alle toewijzingsregels uit de hoofdstukken 2, 3 en 4 verfijningen zijn van deze niet-reduceerbare core. Onder andere introduceren we vanuit Kruskal's algoritme de gelijke-verplichtingen regel. Zowel de niet-reduceerbare core als de gelijke-verplichtingen regel worden axiomatisch gekarakteriseerd. Voor deze karakteriseringen is het handig niet slechts minimum kosten opspannende boomproblemen te behandelen, maar ook problemen waarin er al een partieel netwerk ligt, dat uitgebreid moet worden tot een netwerk waarin alle gebruikers met de bron verbonden zijn. Deze problemen noemen we minimum kosten opspannende uitbreidingsproblemen. Het bestuderen van minimum kosten opspannende uitbreidingsproblemen staat toe de oplossing van een probleem te vergelijken met de oplossing van het partiële probleem verkregen door het algoritme halverwege af te breken.

Hoofdstuk 4 introduceert de proportionele toewijzingsregel, die met Kruskal's algoritme samenhangt, en de gedecentraliseerde regel, die met Borůvka's algoritme samenhangt. Beide regels hebben gemeen dat ze bij iedere kant die geconstrueerd wordt bepalen welke gebruikers verplichtingen hebben met betrekking tot de aanleg van deze kant en dat de kosten van de kant proportioneel met deze verplichtingen verdeeld worden. De proportionele regel wordt axiomatisch gekarakteriseerd.

Hoofdstuk 5 presenteert andere netwerkconstruktiemodellen met vooral suggesties voor verder onderzoek. Het eerste model veronderstelt dat gebruikers niet noodzakelijk met de bron verbonden hoeven te zijn, maar het wel willen als dat hun welzijn bevordert. Het tweede model veronderstelt dat er meerdere (onbetrouwbare) bronnen zijn, en dat een gebruiker met een aantal bronnen verbonden moet zijn. Het derde model veronderstelt dat de kosten van een verbinding afhankelijk zijn van het aantal mensen dat deze verbinding gebruikt. De vierde paragraaf introduceert een alternatief niet-coöperatief spel geassociëerd met netwerk constructie problemen.

Deel II bestudeert de invloed van controle uitgeoefend door spelers over economische hulpbronnen op de verdeling van de opbrengst die door samenwerking en gebruik van die hulpbronnen te bereiken is. De hulpbronnen kunnen verschillende vormen aannemen, zoals delen van een netwerk, grondstoffen van een produktie-economie, stukken land, enz.

De vraag is hoe men de opbrengst moet verdelen als men niet alleen met de economische mogelijkheden maar ook met de controlestruktuur rekening wil houden. Net zoals in deel I worden deze situaties geanalyseerd met behulp van coöperatieve spelen.

In hoofdstuk 6 is de onderliggende economische situatie een lineaire produktie-economie, met transport van grondstoffen, afgewerkte produkten en technologieën tussen de verschillende produktieplaatsen. Hier zijn de gecontroleerde hulpbronnen de grondstoffen. Er wordt aangetoond dat de geassocieerde spelen gebalanceerd zijn, en dat op een efficiënte wijze een core-element te berekenen is.

Hoofdstuk 7 behandelt een situatie waar delen van een netwerk opbrengst kunnen genereren, en waar de knooppunten en verbindingen van het netwerk gecontroleerd worden door de spelers. Drie verschillende toewijzingsregels van de opbrengst, verwant met de Shapley-waarde, worden gedefinieerd en axiomatisch gekarakteriseerd.

Hoofdstuk 8 generaliseert de opzet van de twee voorgaande hoofdstukken, door een abstracte situatie te beschouwen, waarin spelers controle uitoefenen op hulpbronnen, waarmee opbrengst gegenereerd kan worden. Eigenschappen van een geassociëerd coöperatief spel worden geanalyseerd.

De controle in de hoofdstukken 6, 7, en 8 wordt uitgeoefend door middel van zogenaamde simpele spelen. De studie naar de overerving van eigenschappen van de economische situaties door de geassocieerde spelen leidde tot een studie van simpele spelen. Uit die studie kwamen nieuwe axiomatische karakteriseringen van de Shapley waarde en de Banzhaf waarde voort. Deze worden gepresenteerd in hoofdstuk 9.

Tenslotte presenteert hoofdstuk 10 een simpel maar efficiënt algoritme om de kernel en nucleolus van spelen met vetospelers te berekenen. Een vetospeler van een spel is een speler wiens afwezigheid uit een coalitie impliceert dat die coalitie opbrengst nul heeft. Zulke spelers duiken op in vele economische situaties, bijvoorbeeld in een markt met een monopolist.